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APPROXIMATION OF SIGNALS (FUNCTIONS) BELONGING TO $Lip\,(\xi\,(t),\,r)-$CLASS BY $C^1\,.\,N_p$ SUMMABILITY METHOD OF CONJUGATE SERIES OF ITS FOURIER SERIES

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Abstract. Recently, Lal [9] has determined the degree of approximation of function belonging to Lip α and weighted classes using Product $C^1\,.\,N_p$ summability with non-increasing weights$\{p_n\}$. In this paper, we determine the degree of approximation of function $\widetilde{f}$, conjugate to a 2π-periodic function f belonging to $Lip\,(\xi\,(t),\,r)-$class using semi-monotonicity on the generating sequence $\{p_n\}$ with proper set of conditions. Few applications of approximation of functions will also be highlighted.

1. Introduction

The method of summability considered here was first introduced by Woronoi [20]. Summability techniques were also applied on some engineering problems like, Chen and Jeng [3] implemented the Cesàro sum of order $(C, 1)$ and$(C, 2)$, in order to accelerate the convergence rate to deal with the Gibbs phenomenon, for the dynamic response of a finite elastic body subjected to boundary traction. Chen et al. [2] applied regularization with Cesàro sum technique for the derivative of the double layer potential. Similarly, Chen and Hong [1] used Cesàro sum regularization technique for hyper singularity of dual Integral equation.

The degree of approximation of functions belonging to $Lip\,\alpha$, $Lip\,(\alpha,\,r)$, $Lip\,(\xi(t),\,r)$ and weighted classes by Nörlund (N_p) matrices and general summability matrices has been proved by various investigators like Govil [5], Khan [7], Qureshi [18], Mohapatra and Chandra [15], Leindler [8], Rhoades et al. [19], Guven and Israfilov [4] and Mishra et al.[[10]-[12]]. Here, Lal [9] has assumed monotonicity on the generating sequence $\{p_n\}$ to prove their theorems.
The approximation of function $\widetilde{f}$, conjugate to a periodic function $f \in Lip\,(\xi\,(t),\,r)$ $(r \geq 1)$ using product $\left(C^1\,.\,N_p\right) -$ summability has not been studied so far. In this paper,

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we obtain a new theorem on the degree of approximation of function $\widetilde{f}$, conjugate to a periodic function $f \in \; Lip\,(\xi\,(t),\, r) \;\; (r \,\geq\, 1) -$ class using semi-monotonicity on the generating sequence $\{p_n\}$ and a proper set of the conditions.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of n^{th} partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let us write

$$\mathrm{P}_n = \sum_{k=0}^{n} p_k \neq 0 \;\; \forall \; n \geq 0, p_{-1} = 0 = \mathrm{P}_{-1} \, and \;\; P_n \; \rightarrow \infty \, asn \rightarrow \infty \, .$$

The sequence to sequence transformation $t_n^N \;=\; \sum_{\nu=0}^{n} p_{\,n-\nu}\, s_\nu / P_n$ defines the sequence $\left\{t_n^N\right\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable N_p to the sum s if $\lim_{n\rightarrow\infty} t_n^N$ exists and is equal to s. In the special case in which

$$p_n \;=\; \begin{pmatrix} n \,+\, \alpha \,-\, 1 \\ \alpha \,-\, 1 \end{pmatrix} \;=\; \frac{(n+\,\alpha)}{(n+1)\,(\alpha)}\,; \;\; (\alpha \,>\, -1)\,.$$

The Nörlund summability N_p reduces to the familiar C^α summability. The product of C^1 summability with a N_p summability defines $C^1 \,.\, N_p$ summability.

Thus the $C^1 \,.\, N_p$mean is given by $t_n^{C\,N} \;=\; \frac{1}{n+1} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} \; p_{k-\nu}\, s_\nu\,.$ If $t_n^{C\,N} \rightarrow s \;$ as $n \,\rightarrow\, \infty$ then the infinite series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable $C^1 \,.\, N_p$ to the sum s if $\lim_{n\rightarrow\infty} t_n^{C\,N}$ exists and is equal to s.

$s_n \;\rightarrow\; s \;\Rightarrow\; N_p\,(s_n) \;=\; t_n^N \;=\; P_n^{-1} \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu \;\rightarrow\; s,\;$ as $n \;\rightarrow\; \infty\,,\; N_p$ method is regular,

$\Rightarrow\; C^1\,(N_p\,(s_n)\,) \;=\; t_n^{C\,N} \;\;\rightarrow\; s,\;$ as $n \;\rightarrow\; \infty\,,\; C^1$ method is regular, $\Rightarrow\; C^1 \,.\, N_p$ method is regular.

Let $L_{2\,\pi}$ be the space of all $2\,\pi$- periodic and Lebesgue integrable functions over $[-\pi,\, \pi]$.
Then the Fourier series of $f \in\; L_{2\,\pi}$ at x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\, x + b_n \,sin\, n\, x) \;\equiv\; \sum_{n=0}^{\infty} A_n\,(x) \tag{1.1}$$

with n^{th} partial sum $s_n(f;\, x)$, where a_n and b_n are the Fourier coefficients of f. The conjugate series of Fourier series is given by

$$\sum_{n=1}^{\infty} (b_n \;\cos n\, x - a_n \sin n\, x) \equiv \; \sum_{n\,=\,1}^{\infty} B_n\,(x)\,. \tag{1.2}$$

A function $f(x) \in Lip\,\alpha$ if $|\, f(x+t) - f(x)\,| = \mathrm{O}\,(|\,t|^\alpha) \;$ for $\; 0 < \alpha \leq 1,\; t > 0$ and $f(x) \in Lip\,(\alpha,\, r),\;$ [7] for $0 \leq x \leq 2\pi,$ if $\|\, f(x+t) - f(x)\,\|_r = \left(\int_0^{2\pi} |\, f(x+t) - f(x)\,|^{\;r} dx\right)^{1/r} =$ $\mathrm{O}\,(|t|^\alpha)\,, \quad 0 < \alpha \leq 1,\; r \geq 1,\; t > 0.$

A signal $f(x) \in Lip(\xi(t), r)$ if

$$\| f(x+t) - f(x) \|_r = \left(\int_0^{2\pi} |f(x+t) - f(x)|^{\,r} dx \right)^{1/r} = O\left(\xi(t)\right), \quad r \geq 1, \ t > 0.$$

If $\xi(t) = t^{\alpha}$ then $Lip\,(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ reduces to the class $Lip\,\alpha$.
Thus, we observe that

$$Lip(\xi(t), r) \overset{\xi(t)=t^{\alpha}}{\longrightarrow} Lip(\alpha, r) \overset{r\to\infty}{\longrightarrow} Lip\,\alpha \text{ for } 0 < \alpha \leq 1, \ r \geq 1, \ t > 0.$$

L_∞- norm of a function $f : R \to R$ is defined by $\| f \|_\infty = \sup \{| f(x) | : x \in R\}$.
L_r- norm of a function is defined by $\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r \, dx \right)^{1/r}, \ r \geq 1$.
The degree of approximation of a function $f : R \to R$ by trigonometric polynomial t_n of order n under *sup* norm $\| \|_\infty$ is defined by ([21]) $\| t_n - f \|_\infty = sup \{| t_n(x) - f(x) | : x \in R\}$ and $E_n(f)$ of a function $f \in L_r$ is given by $E_n(f) = \min_n \| t_n - f \|_r$. The conjugate function $\widetilde{f}(x)$ is defined for almost every x by

$$\widetilde{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot t/2 \, dt = \lim_{h \to 0} \left(-\frac{1}{2\pi} \int_h^{\pi} \psi(t) \cot t/2 \, dt \right).$$

We note that t_n^N and t_n^{CN} are also trigonometric polynomials of degree (or order) n.

Abel's Transformation: The formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k (v_k - v_{k+1}) - U_{m-1} v_m + U_n v_n, \tag{1.3}$$

where $0 \leq m \leq n$, $U_k = u_0 + u_1 + u_2 + \ldots + u_k$, if $k \geq 0$, $U_{-1} = 0$, which can be verified, is known as Abel's transformation and will be used extensively in what follows. If $v_m, v_{m+1}, \ldots, v_n$ are non-negative and non-increasing, the left hand side of (1.3) does not exceed $2\, v_m \max_{m-1 \leq k \leq n} |U_k|$ in absolute value. In fact,

$$\left| \sum_{k=m}^{n} u_k v_k \right| \leq max\, |U_k| \left\{ \sum_{k=m}^{n-1} (v_k - v_{k+1}) + v_m + v_n \right\}$$

$$= 2\, v_m \, max\, |U_k| \,. \tag{1.4}$$

We write throughout $\phi(t) = f(x+t) - 2f(x) + f(x-t)$,

$$W_n = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} (\nu+1) \, |p_\nu - p_{\nu-1}|,$$

$$\tilde{J}(n,t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k - v + 1/2)\,t}{\sin t/2} \tag{1.5}$$

$\tau = [1/t]$, where τ denotes the greatest integer not exceeding $1/t$. Furthermore, C denotes an absolute positive constant, not necessarily the same at each occurrence.

2. Main Theorem

It is well known that the theory of approximations i.e., TFA, which originated from a well-known theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis [16], in general and in Digital Signal Processing [17] in particular, in view of the classical Shannon sampling theorem. Mittal et. al.
([[13], [14]]) have obtained many interesting results on TFA using summability methods without monotonicity on the rows of the matrix T: a digital filter.

Broadly speaking, signals are treated as functions of one variable and images are represented by functions of two variables. But till now, nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function using $C^1 \,.\, N_p$ product summability method of its conjugate series of Fourier series.

Therefore, the purpose of present paper is to establish a quite new theorem on degree of approximation of a function $\tilde{f}\,(x)$, conjugate to a $2\pi-$ periodic function f belonging to $Lip\,(\xi(t),\ r)$, $(r\ \geq 1)-$ class by $C^1 \,.\, N_p$ means of conjugate series of Fourier series using semi-monotonicity on the generating sequence $\{p_n\}$and a proper set of conditions. We prove

Theorem 2.1. *If $\tilde{f}\,(x)$,conjugate to a $2\pi -$ periodic function f belonging to $Lip\,(\xi(t),\ r)$ class, then its degree of approximation by $C^1 \,.\, N_p$ means of conjugate series of Fourier series (1.2) is given by*

$$\left\| \tilde{t}_n^{C\,N} - \tilde{f} \right\|_r = \mathrm{O}\left((n)^{1/2\,r}\, \xi \left(\frac{1}{\sqrt{n}} \right) \right), \tag{2.1}$$

provided $\{p_n\}$ satisfies the

$$W_n \,<\, C, \tag{2.2}$$

and $\xi\,(t)$ satisfies the following conditions: $\{\xi\,(t)/t\}$ is non-increasing in

$$'t'\,, \tag{2.3}$$

$$\left(\int_0^{\pi/\sqrt{n}} \left(\frac{|\,\psi\,(t)\,|}{\xi\,(t)} \right)^r \, dt \right)^{1/r} = \mathrm{O}\,(1)\,, \tag{2.4}$$

$$\left(\int_{\pi\,/\sqrt{n}}^{\pi} \left(\frac{t^{-\delta}\ \ |\,\psi\,(t)\,|}{\xi\,(t)} \right)^r dt \right)^{1/r} = \mathrm{O}\left((\sqrt{n})^{\delta} \right), \tag{2.5}$$

where δ is an arbitrary number such that $s\,(1-\delta)-1>0$, $r^{-1}+s^{-1}=1,\ \ 1\leq r\leq \infty$, conditions (2.4) and (2.5) hold uniformly in x.

Note 2.2. $\xi\left(\frac{\pi}{\sqrt{n}}\right) \leq \pi\,\xi\left(\frac{1}{\sqrt{n}}\right),\ \ \text{for}\ \left(\frac{\pi}{\sqrt{n}}\right) \geq \left(\frac{1}{\sqrt{n}}\right).$

Note 2.3. *Condition $W_n \,<\, C \ \Rightarrow\ n\,p_n \,<\, C\,P_n$, ([6]).*

Note 2.4. *The product transform $C^1 . N_p$ plays an important role in signal theory as a double digital filter [10] and theory of Machines in Mechanical Engineering [10].*

We need the following lemmas for the proof of our theorem:

Lemma 3.1. $\left|\tilde{J}(n,t)\right| = \mathrm{O}[1/t]$ for $0 < t \leq \pi/\sqrt{n}$.

Proof. For $0 < t \leq \pi/\sqrt{n}$, $\sin(t/2) \geq (t/\pi)$ and $|\cos nt| \leq 1$.

$$\left|\tilde{J}(n,t)\right| = \left|\frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{\cos(k-v+1/2)t}{\sin t/2}\right|$$

$$\leq \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{|\cos(k-v+1/2)t|}{|\sin t/2|}$$

$$\leq \tfrac{1}{2t(n+1)}\textstyle\sum_{k=0}^{n} P_k^{-1}\sum_{v=0}^{k} p_\nu$$
$$= \tfrac{1}{2t(n+1)}\textstyle\sum_{k=0}^{n} P_k^{-1} P_k$$
$$= O[\tau].$$

This completes the proof of Lemma 3.1.

Lemma 3.2. Let $\{p_n\}$ *be a non-negative sequence and satisfies (3.2), then*

$$\left|\tilde{J}(n,t)\right| = \mathrm{O}(\tau) + \mathrm{O}\left(\tfrac{\tau^2}{n}\right)\left(\textstyle\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|\right) \text{ uniformly in}$$

$$0 < t \leq \pi. \tag{3.1}$$

Proof. We have

$$\tilde{J}(n,t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{\cos(k-v+1/2)t}{\sin t/2}$$

$$= \frac{1}{2\pi(n+1)}\left(\sum_{k=0}^{\tau-1} + \sum_{k=\tau}^{n}\right) P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{\cos(k-v+1/2)t}{\sin t/2}$$

$$= \tilde{J}_1(n,t) + \tilde{J}_2(n,t), \tag{3.2}$$

say,
where

$$\left|\tilde{J}_1(n,t)\right| = \left|\frac{1}{2\pi(n+1)}\sum_{k=0}^{\tau-1} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{\cos(k-v+1/2)t}{\sin t/2}\right|$$

$$\leq \frac{1}{2\pi(n+1)}\sum_{k=0}^{\tau-1} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{|\cos(k-v+1/2)t|}{|\sin t/2|} \leq \frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1} P_k^{-1}\sum_{v=0}^{k} p_\nu$$

$$= \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right), \tag{3.3}$$

and using Abel's transformation and $\sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$, *we get*

$$\left|\tilde{J}_2(n,t)\right| = \left|\frac{1}{2\pi(n+1)}\sum_{k=\tau}^{n} P_k^{-1}\sum_{v=0}^{k} p_\nu \frac{\cos(k-v+1/2)t}{\sin t/2}\right|$$

$$\leq \frac{1}{2\,t\,(n+1)} \sum_{k=\tau}^{n} P_{k}^{-1} \left\{ \sum_{v=0}^{k-1} |\Delta\, p_\nu| \; \left| \left(\sum_{\gamma=0}^{\nu} \cos\left(k - \gamma + 1/2\right) t \right) \right| + \left| \left(\sum_{\gamma=0}^{k} \cos\left(k - \gamma + 1/2\right) t \right) \right| p_k \right\}$$

$$= \frac{\mathrm{O}\left(t^{-1}\right)}{2\,t\,(n+1)} \left(\sum_{k=\tau}^{n} P_{k}^{-1} \sum_{v=0}^{k-1} |\Delta\, p_\nu| \; + \sum_{k=\tau}^{n} P_{k}^{-1}\, p_k \right)$$

by virtue of the fact that $\sum_{k=\lambda}^{\mu} \exp\,(-ikt) = \mathrm{O}\left(t^{-1}\right)$, $0 \leq \lambda \leq k \leq \mu$.

$$\left|\tilde{J}_2\,(n,t)\right| = \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right) \left(\sum_{k=\tau}^{n} P_{k}^{-1} \sum_{v=0}^{k-1} |\Delta\, p_\nu| \; + \sum_{k=\tau}^{n} P_{k}^{-1}\, p_k \frac{k}{k} \right)$$

$$= \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right) \left(\sum_{k=\tau}^{n} P_{k}^{-1} \sum_{v=0}^{k-1} |\Delta\, p_\nu| \; + \frac{(n+1)}{\tau} \right)$$

$$= \mathrm{O}\,(\tau) + \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right) \sum_{k=\tau}^{n} P_{k}^{-1} \sum_{v=0}^{k-1} |\Delta\, p_\nu| \,,$$

$$\left|\tilde{J}_2\,(n,t)\right| = \mathrm{O}\,(\tau) + \mathrm{O}\left(\frac{\tau^2}{n}\right) \sum_{k=\tau}^{n} P_{k}^{-1} \sum_{v=0}^{k-1} |\Delta\, p_\nu| \tag{3.4}$$

in view of note 2.3. Combining (3.2), (3.3) and (3.4) yields (3.1). This completes the proof of Lemma 3.2.

Proof of Theorem 2.1: Let $\tilde{s}_n\,(f;\ x)$ denotes the partial sum of series (1.2), we have

$$\tilde{s}_n\,(f;\ x) - \tilde{f}\,(x) = \frac{1}{2\pi} \int_0^{\pi} \psi\,(t)\, \frac{\cos\,(n+1/2)\,t}{\sin t/2} dt.$$

Denoting $C^1 . N_p$ means of $\widetilde{s_n}\,(f\,;\,x)$ by $\tilde{t}_n^{C\,N}$, we write

$$\tilde{t}_n^{C\,N}(x) - \tilde{f}\,(x) = \int_0^{\pi} \psi\,(t)\, \frac{1}{2\,\pi\,(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos\,(k - v + 1/2)\,t}{\sin t/2}\; dt$$

$$= \int_0^{\pi} \psi\,(t)\, \tilde{J}\,(n,t)\; dt = \left[\int_0^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi} \right] \psi\,(t)\, \tilde{J}\,(n,t)\; dt$$

$$= I_1 + I_2 \text{ (say).} \tag{4.1}$$

Clearly, $|\psi\,(x+t) - \psi\,(t)| \leq |f\,(u+x+t) - f\,(u+x)| + |f\,(u-x-t) - f\,(u-x)|$. Hence, by Minkowski's inequality,

$$\left\{ \int_0^{2\pi} |(\psi\,(x+t) - \psi\,(t))\,|^{\,r}\, dx \right\}^{1/r} \leq \left\{ \int_0^{2\pi} |(f\,(u+x+t) - f\,(u+x))\,|^{\,r}\, dx \right\}^{1/r} + \left\{ \int_0^{2\pi} |(f\,(u-x-t) - f\,(u-x))\,|^{\,r}\, dx \right\}^{1/r} = \mathrm{O}\;(\xi\;(t))\,.$$

Then $f \in Lip\,(\xi\,(t)\,,\,r) \Rightarrow \psi \in Lip\,(\xi\,(t)\,,\,r)$. Using Hlder's Inequality, $\psi\,(t) \in Lip\,(\xi\,(t)\,,\,r)$, condition (2.4) , $\sin{(t/2)} \geq (t/\pi)$, for $0 < t \leq \pi$, Lemma 3.1, note 2.2 and Second Mean Value Theorem for integrals, we have

$$|I_1| \leq \left[\int_0^{\pi/\sqrt{n}} \left(\frac{|\,\psi(t)\,|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_0^{\pi/\sqrt{n}} \left(\xi(t)\left|\tilde{J}(n,t)\right|\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\,(1)\ \left[\int_0^{\pi/\sqrt{n}} \left(\frac{\xi(t)}{t}\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left\{\xi\left(\frac{\pi}{\sqrt{n}}\right)\right\}\left[\int_h^{\pi/\sqrt{n}} \left(\frac{1}{t}\right)^s dt\right]^{1/s}, \text{ as } h \to 0$$

$$= \mathrm{O}\left((n)^{r/2}\ \xi\left(\frac{1}{\sqrt{n}}\right)\right), \quad r^{-1} + s^{-1} = 1. \tag{4.2}$$

Using Lemma 3.2, we have

$$|I_2| = \mathrm{O}\left[\int_{\pi/\sqrt{n}}^{\pi} \frac{|\psi\,(t)\,|}{t}\,dt\right] + \mathrm{O}\left[\int_{\pi/\sqrt{n}}^{\pi} \frac{|\psi\,(t)\,|}{t\,n}\left(\tau\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\,p_\nu|\right) dt\right]$$
$$= \mathrm{O}\,(I_{2\,1}) + \mathrm{O}\,(I_{2\,2})\,.$$

Using Hlder's Inequality, conditions (2.3) and (2.5), note 2.2 and Second Mean Value Theorem for integrals, we have

$$|I_{2\,1}| \leq \left[\int_{\pi/\sqrt{n}}^{\pi} \left(\frac{t^{-\delta}\,|\,\psi\,(t)\,|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_{\pi/\sqrt{n}}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+1}}\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left((\sqrt{n})^{\delta}\right)\left[\int_{\pi/\sqrt{n}}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+1}}\right)^s dt\right]^{1/s} = \mathrm{O}\left\{(\sqrt{n})^{\delta}\right\}\left[\int_{1/\pi}^{\sqrt{n}/\pi} \left(\frac{\xi\left(1/y\right)}{y^{\delta-1}}\right)^s \frac{dy}{y^2}\right]^{1/s}$$

$$= \mathrm{O}\left((\sqrt{n})^{\delta}\,\frac{\xi\left(\pi/\sqrt{n}\right)}{\pi/\sqrt{n}}\right)\left[\int_{1/\pi}^{\sqrt{n}/\pi} \left(\frac{dy}{y^{\,\delta\,s\,+\,2}}\right)\right]^{1/s}$$

$$= \mathrm{O}\left((\sqrt{n})^{\delta+1}\,\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{(\sqrt{n})^{-\delta\,s-1} - (\pi)^{\delta\,s+1}}{-\delta\,s-1}\right)^{1/s}$$

$$= \mathrm{O}\left((n)^{r/2}\ \xi\left(\frac{1}{\sqrt{n}}\right)\right). \tag{4.3}$$

Similarly, as above conditions (2.2), (2.3), (2.5), note 2.2 and Second Mean Value Theorem for integrals, we have

$$|I_{2\,2}| \leq \left[\int_{\pi/\sqrt{n}}^{\pi} \left(\frac{t^{-\delta}\,|\,\psi\,(t)\,|}{\xi(t)}\right)^r dt\right]^{1/r}$$

$$\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\frac{1}{n}\ \left(\tau\sum_{k=\tau}^{n}P_k^{-1}\ \sum_{\nu=0}^{k-1}|\,p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\ \left(\tau\sum_{k=\tau}^{n}P_k^{-1}\ \sum_{\nu=0}^{k-1}|\,p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\ \left(\sum_{k=\tau}^{n}P_k^{-1}\ \sum_{\nu=0}^{k-1}(\nu\ +1)\,|\,p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\ \left(\sum_{k=0}^{n}P_k^{-1}\ \sum_{\nu=0}^{k}(\nu\ +1)\,|\,p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\,W_n\ 2\,\pi\,(n)\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left((\sqrt{n})^{\delta}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}}\right)^s dt\right]^{1/s}$$

$$= \mathrm{O}\left\{(\sqrt{n})^{\delta}\right\}\left[\int_{1/\pi}^{\sqrt{n}/\pi}\left(\frac{\xi\left(1/y\right)}{y^{\delta-1}}\right)^s \frac{dy}{y^2}\right]^{1/s}$$

$$= \mathrm{O}\left((n)^{r/2}\,\xi\left(\frac{1}{\sqrt{n}}\right)\right)\ . \tag{4.4}$$

Collecting (4.1)-(4.4), we have

$$\left|\,\tilde{t}_n^{C\,N}-\widetilde{f}\,\right| = \mathrm{O}\left((n)^{\,r/2}\,\xi\ \left(\frac{1}{\sqrt{n}}\right)\right). \tag{4.5}$$

Now, using the L_r-norm of a function, we get

$$\begin{aligned}\left\|\,\tilde{t}_n^{C\,N}-\widetilde{f}\,\right\|_r &= \left\{\int_0^{2\pi}\left|\tilde{t}_n^{C\,N}\,(x)-\widetilde{f}\,(x)\right|^{\mathrm{r}}\mathrm{dx}\right\}^{1/r}\\ &= \mathrm{O}\left(\int_0^{2\pi}\left((n)^{\,r/2}\,\xi\ \left(\tfrac{1}{\sqrt{n}}\right)\right)^r\mathrm{dx}\right)^{1/r}\end{aligned}$$

$$= \mathrm{O}\left((n\,)^{r/2}\ \ \xi\ \left(\frac{1}{\sqrt{n}}\right)\ \left(\int_0^{2\pi}\mathrm{dx}\right)^{1/r}\right)$$

$$= \mathrm{O}\ \left((n\,)^{r/2}\ \ \xi\ \left(\frac{1}{\sqrt{n}}\right)\right)\ .$$

This completes the proof of Theorem 2.1.

3. Applications

Some interesting applications of the Cesàro summability can be seen [[1], [2], [3]]. The following corollaries can be derived from Theorem 2.1.

Corollary 5.1. *If* $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, then the class $Lip\,(\xi(t), r)$, $r \geq 1$, reduces to the class $Lip\,(\alpha, r)$, $\frac{1}{r} < \alpha < 1$ and the degree of approximation of a function $\widetilde{f}\,(x)$, conjugate to a $2\pi-$ periodic function f belonging to the class $Lip\,(\alpha, r)$, is given by

$$\left| \tilde{t}_n^{C\,N} - \widetilde{f} \right| = \mathrm{O}\left((n)^{-\alpha/2 + 1/2r} \right). \tag{5.1}$$

Proof. Putting $\xi(t) = t^{\alpha},\ 0 < \alpha \leq 1$ in Theorem 2.1, we have

$$\left\| \tilde{t}_n^{C\,N} - \widetilde{f} \right\|_r = \left\{ \int_0^{2\pi} \left| \tilde{t}_n^{C\,N}(x) - \widetilde{f}\,(x) \right|^{\mathrm{r}} \mathrm{dx} \right\}^{1/r} = \mathrm{O}\left((n)^{r/2}\ \xi\left(1/\sqrt{n}\right) \right)$$

$$= \mathrm{O}\left((n)^{-\alpha/2 + r/2} \right).$$

Thus we get

$$\left| \tilde{t}_n^{C\,N} - \widetilde{f} \right| \leq \left\{ \int_0^{2\pi} \left| \tilde{t}_n^{C\,N}(x) - \widetilde{f}\,(x) \right|^{\mathrm{r}} \mathrm{dx} \right\}^{1/r} = \mathrm{O}\left((n)^{-\alpha/2 + r/2} \right),\ \ r \geq 1.$$

This completes the proof of corollary 5.1.

Corollary 5.2. If $\xi\,(t) = t^{\alpha} for\ 0 < \alpha < 1 and\ r \to \infty$ in (5.1), then $f \in Lip\,\alpha$. In this case, using (5.1), we have

$$\left\| \widetilde{f}(x) - \tilde{t}_n^{C\,N}(x) \right\|_{\infty} = \mathrm{O}\left((n)^{-\alpha/2} \right). \tag{5.2}$$

Proof. For $r \to \infty$, we get

$$\left\| \widetilde{f}(x) - \tilde{t}_n^{C\,N}(x) \right\|_{\infty} = \sup_{0 \leq x \leq 2\pi} \left| \widetilde{f}(x) - \tilde{t}_n^{C\,N}(x) \right|_r = \mathrm{O}\left((n)^{-\alpha/2} \right).$$

This completes the proof of corollary 5.2.

4. Conclusion

Various results concerning to the degree of approximation of periodic signals (functions) belonging to the *Lip* $(\xi(t), r)$, $(r = 1)$-class by Matrix Operator have been reviewed and the condition of monotonicity on the weights $\{p_n\}$ has been relaxed (i.e. weakening the conditions on the filter, we improve the quality of digital filter). Further, a proper (correct) set of conditions have been discussed to rectify the errors. Some interesting application of the operator $(C^1.\ N_p)$ used in this paper pointed out in Note 2.4.

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