Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 4(2012), Pages 35-44.

SUBCLASSES OF STARLIKE FUNCTIONS INVOLVING SRIVASTAVA-ATTIYA INTEGRAL OPERATOR

(COMMUNICATED BY INDRAJIT LAHIRI)

SHIGEYOSHI OWA [1,*] G.MURUGUSUNDARAMOORTHY [2] AND N.MAGESH[3]

Abstract. Making use of the generalized Srivastava-Attiya integral operator, we define a new subclass of starlike functions with negative coefficients and obtain coefficient estimates, extreme points, the radii of close to convexity, starlikeness and convexity and neighbourhood results for $f \in TS_b^{\mu}(\alpha, \beta, \gamma, A, B)$. In particular, we obtain modified Hadamard product results for the function $f(z)$ belongs to the class $TS_b^{\mu}(\alpha, \beta, \gamma, A, B)$ in the unit disc.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C},\ |z| < 1\}$. Also denote by T a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U, \tag{1.2}$$

introduced and studied by Silverman [24]. For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Univalent, starlike, convex, Hadamard product(or Convolution), Hurwitz-Lerch Zeta function Srivastava-Attiya Integral operator.

Submitted May 20, 2011. Published November 9, 2012.

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [27] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}; (a \in \mathbb{C}\backslash\{\mathbb{Z}_0^-\}; s \in \mathbb{C} \text{ when } |z|<1; \Re(s)>1 \text{ and } |z|=1) \tag{1.4}$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\}$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [5], Ferreira and Lopez [8], Garg et al. [10], Lin and Srivastava [12], Lin et al. [13], and others. Srivastava and Attiya [25] (see also Raducanu and Srivastava [21], and Prajapat and Goyal [20]) introduced the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z), (z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A}) \tag{1.5}$$

where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in U). \tag{1.6}$$

We recall here the following relationships (given earlier by [20], [21]) which follow easily by using (1.1), (1.5) and (1.6)

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{1.7}$$

Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [3] introduced the integral operator

$$\mathcal{J}_{\mu,b}^{\lambda,k} f(z) = z + \sum_{n=2}^{\infty} C_n^{\lambda}(b,\mu) a_n z^n, \tag{1.8}$$

where

$$C_n^{\lambda}(b,\mu) = \left| \left(\frac{1+b}{n+b}\right)^{\mu} \frac{\lambda!(n+k-2)!}{(k-2)!(n+\lambda-1)!} \right| \tag{1.9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}, k \geq 2$ and $\lambda > -1$. Further note that $J_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator, and $J_{0,b}^{\lambda,k}$ is the well-known Choi-Saigo- Srivastava operator (see [6, 14]). It is of interest to note that for $\lambda = 1; k = 2$, and specializing the parameters μ and b suitably we get various integral operators introduced by Alexander [2] , Bernardi[4] and Jung-Kim-Srivastava integral operator [15] closely related to some multiplier transformation studied by Fleet [9]. Making use of the operator $\mathcal{J}_{\mu,b}^{\lambda,k}$, and motivated by the earliear works of Murugusundaramoorthy[17, 18] we introduce a new subclass of analytic functions with negative coefficients and discuss some usual properties of the geometric function theory of this generalized function class.

For fixed $-1 \leq A \leq B \leq 1$ and $0 < B \leq 1$,let $S_b^{\mu}(\alpha,\beta,\gamma,A,B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$\left|\frac{\frac{z(\mathcal{J}_{\mu,b}^{\lambda,k}f(z))'}{\mathcal{J}_{\mu,b}^{\lambda,k}f(z)}-1}{2\gamma(B-A)\left(\frac{z(\mathcal{J}_{\mu,b}^{\lambda,k}f(z))'}{\mathcal{J}_{\mu,b}^{\lambda,k}f(z)}-\alpha\right)-B\left(\frac{z(\mathcal{J}_{\mu,b}^{\lambda,k}f(z))'}{\mathcal{J}_{\mu,b}^{\lambda,k}f(z)}-1\right)}\right|<\beta,\quad z\in U \tag{1.10}$$

where $\mathcal{J}_{\mu,b}^{\lambda,k}f(z)$ is given by (1.8), $0\leq\alpha<1$, $0<\beta\leq1$, $\frac{B}{2(B-A)}<\gamma\leq\begin{cases}\frac{B}{2(B-A)\alpha} & \alpha\neq0,\\ 1 & \alpha=0.\end{cases}$
We also let $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)=S_b^{\mu}(\alpha,\beta,\gamma,A,B)\cap T$.

For convenience in entire paper we consider $0\leq\alpha<1$, $0<\beta\leq1$,

$$\frac{B}{2(B-A)}<\gamma\leq\begin{cases}\frac{B}{2(B-A)\alpha} & \alpha\neq0,\\ 1 & \alpha=0.\end{cases}$$

for fixed $-1\leq A\leq B\leq1$ and $0<B\leq1$, one or otherwise stated.

By suitably specializing the values of A,B,α,β and γ the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$ leads to known subclasses studied in [1, 16] and [19] and various new subclasses.

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bound, extreme points, radii of close to convexity, starlikeness and convexity for the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$. Further, we obtain modified Hadamard product and Neighbourhood results for aforementioned class.

2. Characterization Properties

We now obtain the characterization property for functions $f(z)$ to belong to the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$ there by obtaining coefficient bounds.

Theorem 2.1. *Let the function $f(z)$ be defined by (1.2) is in the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$ if and only if*

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n^{\lambda}(b,\mu)|a_n|\leq2\beta\gamma(1-\alpha)(B-A), \tag{2.1}$$

where $C_n^{\lambda}(b,\mu)$ is given by (1.9).

Proof. The proof of Theorem 2.1 is much akin to the proof of theorems on coefficient bounds established in [7, 17, 26], so we skip the details in this regard. □

Corollary 2.2. *Let the function $f(z)$ defined by (1.2) be in the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$. Then we have*

$$a_n\leq\frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n^{\lambda}(b,\mu)} \tag{2.2}$$

The equation (2.2) is attained for the function

$$f(z)=z-\frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]C_n^{\lambda}(b,\mu)}z^n\qquad(n\geq2) \tag{2.3}$$

where $C_n^{\lambda}(b,\mu)$ is given by (1.9).

For the sake of brevity , we let

$$\Phi_n(\alpha,\beta,\gamma,A,B)=[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)] \tag{2.4}$$

and

$$\Phi_2(\alpha,\beta,\gamma,A,B) = [1 + 2\beta\gamma(B-A)(2-\alpha) - B\beta] \tag{2.5}$$

unless otherwise stated.

Theorem 2.3. *Let the function $f(z)$ defined by (1.2) belong to $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$. Then*

$$|f(z)| \geq |z| \left\{1 - \frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^{\lambda}(b,\mu)}\ |z|\right\} \tag{2.6}$$

and

$$|f(z)| \leq |z| \left\{1 + \frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^{\lambda}(b,\mu)}\ |z|\right\}, \tag{2.7}$$

where

$$C_2^{\lambda}(b,\mu) = \left(\frac{1+b}{2+b}\right)^{\mu} \frac{(k-1)k}{(1+\lambda)}. \tag{2.8}$$

Proof. In the view of (2.1) and the fact that $C_n^{\lambda}(b,\mu)$ is non-decreasing for $n \geq 2, 0 \leq \alpha < 1$ we have

$$[2\beta\gamma(B-A)(2-\alpha) + (1-B\beta)]C_2^{\lambda}(b,\mu)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \Phi_n(\alpha,\beta,\gamma,A,B)C_n^{\lambda}(b,\mu)a_n$$
$$\leq \quad 2\beta\gamma(1-\alpha)(B-A)$$

which readily yields,

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[1 + 2\beta\gamma(B-A)(2-\alpha) - B\beta)]C_2^{\lambda}(b,\mu)}. \tag{2.9}$$

Theorem 2.3 follows readily from (1.2) and (2.9). □

Theorem 2.4. *(Extreme Points) Let$f_1(z) = z; f_n(z) = z - \frac{2\beta\gamma(1-\alpha)(B-A)}{\Phi_n(\alpha,\beta,\gamma,A,B)C_n^{\lambda}(b,\mu)} z^n, (n \geq 2)$ where $C_n^{\lambda}(b,\mu)$ is given by (1.9). Then $f(z)$ is in the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$ where $\omega_n \geq 0$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} \omega_n = 1$.*

We shall prove the following results for the closure of functions in the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$.

Let the functions $f_j(z)(j = 1, 2)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\ z^n \text{ for } a_{n,\ j} \geq 0,\ z \in U. \tag{2.10}$$

Theorem 2.5. *(Closure Theorem) Let the functions $f_j(z)(j = 1, 2, \dots m)$ defined by (2.10) be in the classes $TS_b^{\mu}(\alpha_j,\beta,\gamma,A,B)$ $(j = 1, 2, \dots m)$ respectively. Then the function $h(z)$ defined by $h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} a_{n,\ j}\right) z^n$ is in the class $TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$, where $\alpha = \min_{1\leq j\leq m}\{\alpha_j\}$ where $0 \leq \alpha_j \leq 1$.*

Proof. Since $f_j \in TS_b^\mu(\alpha_j, \beta, \gamma, A, B)$, $(j = 1, 2, \ldots m)$ by applying Theorem 2.1, to (2.10) we observe that

$$\sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu) \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,\ j} \right) = \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu) a_{n,\ j} \right)$$

$$\leq \frac{1}{m} \sum_{j=1}^{m} 2\beta\gamma(1-\alpha_j)(B-A) \leq 2\beta\gamma(1-\alpha)(B-A)$$

which in view of Theorem 2.1, again implies that $h \in TS_b^\mu(\alpha, \beta, \gamma, A, B)$ and so the proof is complete. □

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$.

Theorem 2.6. *Let the function $f(z)$ defined by (1.2)belong to the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$. Then $f(z)$ is close-to-convex of order σ $(0 \leq \sigma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \inf \left[\frac{(1-\sigma)\Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu)}{2n\beta\gamma(B-A)(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (n \geq 2), \tag{2.11}$$

where $C_n^\lambda(b, \mu)$ is given by (1.9). The result is sharp, with extremal function $f(z)$ given by (2.4).

Proof. Given $f \in T$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma. \tag{2.12}$$

For the left hand side of (2.12) we have $|f'(z) - 1| \leq \sum\limits_{n=2}^{\infty} n a_n |z|^{n-1}$. The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\sigma} a_n |z|^{n-1} < 1,$$

that is, if

$$\frac{n}{1-\sigma} |z|^{n-1} \leq \frac{\Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu)}{2\beta\gamma(B-A)(1-\alpha)}.$$

where we have made use of the assertion (2.1) of Theorem 2.1.The last inequality leadus immediately to the disk $|z| < r_1$ where r_1 given by (2.11,which completes the proof. □

Theorem 2.7. *Let $f \in TS_b^\mu(\alpha, \beta, \gamma, A, B)$. Then*

(i) *f is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_2$; that is, Re $\left\{ \frac{zf'(z)}{f(z)} \right\} > \sigma$, where*

$$r_2 = \inf \left[\left(\frac{1-\sigma}{n-\sigma} \right) \frac{\Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu)}{2\beta\gamma(B-A)(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (n \geq 2) \tag{2.13}$$

(ii) *f is convex of order σ $(0 \leq \sigma < 1)$ in the disc $|z| < r_3$, that is Re $\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \sigma$, where*

$$r_3 = \inf \left[\left(\frac{1-\sigma}{n(n-\sigma)} \right) \frac{\Phi_n(\alpha, \beta, \gamma, A, B) C_n^\lambda(b, \mu)}{2\beta\gamma(B-A)(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (n \geq 2) \tag{2.14}$$

where $C_n^{\lambda}(b,\mu)$ is given by (1.9). Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof. Following the techniques employed in [26],we can easily prove (i)

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii). □

3. **Modified Hadamard Products**

Let the functions $f_j(z)(j=1,2)$ be defined by (2.10). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \; = \; z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}\; z^n.$$

Using the techniques of Schild and Silverman [23], we prove the following results.

Theorem 3.1. *For functions $f_j(z)(j=1,2)$ defined by (2.10), let $f_1 \in TS_b^{\mu}(\alpha,\beta,\gamma,A,B)$ and $f_2 \in TS_b^{\mu}(\delta,\beta,\gamma,A,B)$. Then $(f_1 * f_2) \in TS_b^{\mu}(\xi,\beta,\gamma,A,B)$, where*

$$\xi = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\delta)(1+2\beta\gamma(B-A)-B\beta)}{\Phi_2(\alpha,\beta,\gamma,A,B)\Phi_2(\delta,\beta,\gamma,A,B)C_2^{\lambda}(b,\mu) - 4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\delta)} \tag{3.1}$$

and $\Phi_2(\alpha,\beta,\gamma,A,B)$is given by (2.5) , $C_2^{\lambda}(b,\mu)$ is given by (2.8)and $\Phi_2(\delta,\beta,\gamma,A,B,2) =$ $[2\beta\gamma(B-A)(2-\delta)+(1-B\beta)]$.

Proof. In view of Theorem 2.1, it suffice to prove that

$$\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]C_n^{\lambda}(b,\mu)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \le 1, \;\; (0 \le \xi < 1)$$

where ξ is defined by (3.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}(C_n^{\lambda}(b,\mu))^{-1}} \sqrt{a_{n,1}a_{n,2}} \;\le 1, \tag{3.2}$$

where $\Phi_n(\alpha,\beta,\gamma,A,B)$ is given by (2.4) and $\Phi_n(\delta,\beta,\gamma,A,B,n) = [2\beta\gamma(B-A)(n-\delta)+(1-B\beta)(n-1)]$. Thus we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{[\Phi_n(\xi,\beta,\gamma,A,B)]C_n^{\lambda}(b,\mu)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \le \sum_{n=2}^{\infty} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}(C_n^{\lambda}(b,\mu))^{-1}} \sqrt{a_{n,1}a_{n,2}}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \quad \le \quad \frac{1-\xi}{\sqrt{(1-\alpha)(1-\delta)}} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{[\Phi_n(\xi,\beta,\gamma,A,B)]}, \;\; (n \ge 2)$$

where $\Phi_n(\xi,\beta,\gamma,A,B) = 2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)$. By view of (3.2) it is sufficient to find largest ξ such that

$$\frac{2\beta\gamma(B-A)\sqrt{(1-\alpha)(1-\delta)}(C_n^{\lambda}(b,\mu))^{-1}}{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}} \le \frac{1-\xi}{\sqrt{(1-\alpha)(1-\delta)}} \frac{[\Phi_n(\alpha,\beta,\gamma,A,B)]^{1/2}[\Phi_n(\delta,\beta,\gamma,A,B)]^{1/2}}{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]}$$

which yields

$$\xi = \Psi(n) = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\delta)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{[\Phi_n(\alpha,\beta,\gamma,A,B)\Phi_n(\delta,\beta,\gamma,A,B)]C_n^{\lambda}(b,\mu) - 4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\delta)} \tag{3.3}$$

for $n \geq 2$ is an increasing function of n $(n \geq 2)$ and letting $n = 2$ in (3.3), we get the desired result. □

By using arguments similar to those in proof of Theorem 3.1,and employing the techniques of [26] we can easily prove the following results, hence we state the following theorems without proof.

Theorem 3.2. *Let the functions $f_j(z)(j = 1, 2)$ defined by (2.10), be in the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$ then $(f_1 * f_2) \in TS_b^\mu(\rho, \beta, \gamma, A, B)$, where $\rho = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)^2(1+2\beta\gamma(B-A)-B\beta)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]^2 C_2^\lambda(b,\mu) - 4\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$ and $C_2^\lambda(b, \mu)$ is given by (2.8).*

Proof. By taking $\delta = \alpha$, in the above theorem, the result follows. □

Theorem 3.3. *Let the function $f(z)$ defined by (1.2) be in the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$. Also let $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ for $|b_n| \leq 1$. Then $(f * g) \in TS_b^\mu(\alpha, \beta, \gamma, A, B)$.*

Theorem 3.4. *Let the functions $f_j(z)(j = 1, 2)$ defined by (2.10) be in the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$. Then the function $h(z)$ defined by $h(z) = z - \sum_{n=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$ is in the class $TS_b^\mu(\xi, \beta, \gamma, A, B)$, where $\xi = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)}{C_2^\lambda(b,\mu)[\Phi_2(\alpha,\beta,\gamma,A,B)]^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$ and $C_2^\lambda(b, \mu)$ is given by (2.8).*

4. Inclusion relations involving $N_\delta(e)$

Following [11, 22], we define the $\delta-$ neighbourhood of function $f \in T$ by

$$N_\delta(f) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta \right\}. \tag{4.1}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \leq \delta \right\}. \tag{4.2}$$

Now we obtain inclusion relations of the class $TS_b^\mu(\alpha, \beta, \gamma, A, B)$.

Theorem 4.1. *If*

$$\delta := \frac{4\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)} \tag{4.3}$$

where $C_2^\lambda(b, \mu)$ is given by (2.8). Then $TS_b^\mu(\alpha, \beta, \gamma, A, B) \subset N_\delta(e)$.

Proof. For $f \in TS_b^\mu(\alpha, \beta, \gamma, A, B)$, Theorem 2.1 immediately yields

$$[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu) \sum_{n=2}^{\infty} a_n \leq 2\beta\gamma(1-\alpha)(B-A),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)}. \tag{4.4}$$

On the other hand, from (2.1) and (4.4) that

$$\begin{aligned}
&[2\beta\gamma(B-A)+(1-B\beta)]C_2^\lambda(b,\mu)\sum_{n=2}^{\infty}na_n \\
\leq\;& 2\beta\gamma(1-\alpha)(B-A)+[2\beta\gamma\alpha(B-A)+(1-B\beta)]C_2^\lambda(b,\mu)\times\left[\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)}\right] \\
=\;& \frac{2[2\beta\gamma(1-\alpha)(B-A)][2\beta\gamma(B-A)+(1-B\beta)]}{[\Phi_2(\alpha,\beta,\gamma,A,B)]}
\end{aligned}$$

that is

$$\sum_{n=2}^{\infty}na_n \leq \frac{4\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)} := \delta \tag{4.5}$$

which, in view of the (4.2) which complete the proof of Theorem 4.1. □

Next we determine the neighborhood for the class $TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$ which we define as follows. A function $f\in T$ is said to be in the class $TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$ if there exists a function $h\in TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$ such that $\left|\frac{f(z)}{h(z)}-1\right|<1-\rho,\;(z\in U,\;0\leq\rho<1)$.

Theorem 4.2. *If $h\in TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$ and*

$$\rho = 1-\frac{[\Phi_2(\alpha,\beta,\gamma,A,B)]\delta C_2^\lambda(b,\mu)}{[2+4\beta\gamma(B-A)(2-\alpha)-B\beta]C_2^\lambda(b,\mu)-4\beta\gamma(1-\alpha)(B-A)} \tag{4.6}$$

then $N_\delta(h)\subset TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$.

Proof. Suppose that $f\in N_\delta(g)$ we then find from (4.1) that $\sum_{n=2}^{\infty}n|a_n-b_n|\leq\delta$ which implies that the coefficient inequality $\sum_{n=2}^{\infty}|a_n-b_n|\leq\frac{\delta}{2}$. Since $h\in TS_b^\mu(\alpha,\beta,\gamma,A,B)$, we have $\sum_{n=2}^{\infty}b_n\leq\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)}$ so that

$$\begin{aligned}
\left|\frac{f(z)}{h(z)}-1\right| &< \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n} \leq \frac{\frac{\delta}{2}}{1-\frac{2\beta\gamma(1-\alpha)(B-A)}{[\Phi_2(\alpha,\beta,\gamma,A,B)]C_2^\lambda(b,\mu)}} \\
&= \frac{[\Phi_2(\alpha,\beta,\gamma,A,B)]\delta C_2^\lambda(b,\mu)}{[2+4\beta\gamma(B-A)(2-\alpha)-B\beta]C_2^\lambda(b,\mu)-4\beta\gamma(1-\alpha)(B-A)} = 1-\rho.
\end{aligned}$$

provided that ρ is given precisely by (4.6). Thus by definition, $f\in TS_b^\mu(\rho,\alpha,\beta,\gamma,A,B)$ for ρ given by (4.6), which completes the proof. □

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 2.1 to Theorem 4.2, one can state the corresponding results for many relatively more familiar function classes.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] R. Aghalary and S. Kullkarni, *Some theorems on univalent functions,* J. Indian Acad. Math., 24 (1), (2002), 81–93.

[2] J. W. Alexander,*Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17**(1915), 1222.

[3] K. Al-Shaqsi and M. Darus, *On certain subclasses of analytic functions defined by a multiplier transformation with two parameters*, Appl. Math. Sci. (2009), in press.

[4] S. D. Bernardi,*Convex and starlike univalent functions,* Trans. Amer. Math. Soc., 135 ,(1969), 429–446.

[5] J. Choi and H. M. Srivastava, *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput., **170** (2005), 399-409.

[6] J. H. Choi, M. Saigo and H. M. Srivastava,*Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl. 276 (2002), 432445.

[7] M.Darus, *Unified treatment of certain subclasses of prestarlike functions* ,JIPAM, Volume 6 Issue 5, Article 132(2005).

[8] C. Ferreira and J. L. Lopez, *Asymptotic expansions of the Hurwitz-Lerch Zeta function*, J. Math. Anal. Appl., **298**(2004), 210-224.

[9] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38**(1972), 746765

[10] M. Garg, K. Jain and H. M. Srivastava,*Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17**(2006), 803-815.

[11] A.W.Goodman, *Univalent functions and nonanalytic curves,* Proc.Amer.Math.Soc., (8)(1957), 598-601.

[12] S.-D. Lin and H. M. Srivastava, *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput., **154**(2004), 725-733.

[13] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, *Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17**(2006), 817-827.

[14] Y. Ling and F.-S. Liu, *The Choi-Saigo-Srivastava integral operator and a class of analytic functions*, Appl. Math. Comput. 165 (2005), 613621.

[15] I. B. Jung, Y. C. Kim AND H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176**(1993), 138147.

[16] S. M. Khairanar and Meena More, *Certain family of analytic and univalent functions with normalized copnditions,* Acta Math. Acade. Paeda. Nyire., 24 (2008), 333–344.

[17] G.Murugusundaramoorthy , *Unified class of analytic functions with negative coefficients involving the Hurwitz-Lerch Zeta function* ,Bulletin of Mathematical Analysis and Applications, Volume 1 Issue 3(2009), 71-84.

[18] G.Murugusundaramoorthy, A subclass of analytic functions associated with the Hurwitz-Lerch Zeta function , Hacepetee J Maths (2010) , Vol.39, No.2, 265–272

[19] S. Owa and J. Nishiwaki, *Coefficient estimates for certain classes of analytic functions,* J. Inequal. Pure. Appl. Math., 3(5) Art.72, (2002).

[20] J. K. Prajapat and S. P. Goyal,*Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal., **3**(2009), 129-137.

[21] D. Raducanu and H. M. Srivastava, *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function*, Integral Transform. Spec. Funct., **18**(2007), 933-943.

[22] St.Rucheweyh, *Neighborhoods of univalent functions,* Proc.Amer.Math.Soc., 81 (1981), 521-527.

[23] A. Schild and H. Silverman, *Convolution of univalent functions with negative coefficients,* Annales Univ. Mariae Curie., 29 (1975), 99 - 107.

[24] H. Silverman, *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc. , **51** (1975), 109 - 116.

[25] H. M. Srivastava and A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct., **18**(2007), 207–216.

[26] H. M. Srivastava and M.K.Aouf,*A certain derivative operator and its applications to new class of analytic and multivalent functions with negative coefficientsI and II*,J.Math.Anal.Appl.**171**,1–13(1992); J.Math.Anal.Appl.**192**,673–688(1995)

[27] H. M. Srivastava and J. Choi,*Series associated with the Zeta and related functions*, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.

Shigeyoshi Owa
Corresponding Author, Department of Mathematics,
Kinki University, Higashi-Osaka, Osaka 577-8502 Japan
E-mail address: owa@math.kindai.ac.jp

G.Murugusundaramoorthy
School of Advanced Sciences, VIT University
Vellore - 632014, India.
E-mail address: gmsmoorthy@yahoo.com

N.Magesh
P.G and Research Department of Mathematics
Govt Arts College(Men), Krishnagiri-635001, Tamilnadu, India.
E-mail address: nmagi2000@gmail.com