

## ON RANDERS CHANGE OF MATSUMOTO METRIC

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ABSTRACT. In this paper we study the properties of special  $(\alpha, \beta)$ -metric  $\frac{\alpha^2}{\alpha-\beta} + \beta$ , the Randers change of Matsumoto metric. We find a necessary and sufficient condition for this metric to be of locally projectively flat and we prove the conditions for this metric to be of Berwald and Douglas type.

### 1. INTRODUCTION

The Matsumoto metric is an interesting  $(\alpha, \beta)$ -metric introduced by using gradient of slope, speed and gravity in [5]. This metric formulates the model of a Finsler space. Many authors ([5], [1], [10], etc) have studied this metric by different perspectives. Projectively flat Finsler spaces are regular distance functions with straight geodesics. An extensive study of projectively flat Finsler metrics was taken up by authors [6], [7], [9], [12], [13], [14], [11] and [15]. Another interesting and important class of Finsler spaces is the class of Berwald spaces. Berwald spaces are the Finsler spaces with linear connections. As a generalization of Berwald space S. Basco and M. Matsumoto [2] introduced the notion of a Douglas space. A Douglas space is a Finsler space where the projectively invariant Douglas tensor vanishes.

The purpose of the present paper is to investigate the special  $(\alpha, \beta)$ -metric  $\frac{\alpha^2}{\alpha-\beta} + \beta$  which is considered to be Randers change of Matsumoto metric.

After preliminaries in section 2, we prove the following in section 3:

The  $(\alpha, \beta)$ -metric  $F = \frac{\alpha^2}{\alpha-\beta} + \beta$  is locally projectively flat if and only if

- (i)  $\beta$  is parallel with respect to  $\alpha$ ,
- (ii)  $\alpha$  is locally projectively flat, i.e., of constant curvature.

In section 4, we prove the conditions that the Finsler space  $F^n$  with the metric  $F = \frac{\alpha^2}{\alpha-\beta} + \beta$  is a Berwald space and a Douglas space.

### 2. PRELIMINARIES

Let  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  be a Riemannian metric,  $\beta = b_i y^i$  a 1-form and let  $F = \alpha\phi(s)$ ,  $s = \frac{\beta}{\alpha}$ , where  $\phi = \phi(s)$  is a positive  $C^\infty$  function defined in a neighborhood

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of the origin  $s = 0$ . It is well known that  $F = \alpha\phi(\beta/\alpha)$  is a Finsler metric for any  $\alpha$  and  $\beta$  with  $b = \|\beta\|_\alpha < b_o$  if and only if

$$\phi(s) > 0, \quad (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o). \quad (2.1)$$

By taking  $b = s$ , we obtain

$$\phi(s) - s\phi'(s) > 0, \quad (|s| < b_o).$$

Let  $G^i$  and  $G_\alpha^i$  denote the spray coefficients of  $F$  and  $\alpha$  respectively, given by

$$G^i = \frac{g^{il}}{4} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^k} \}, \quad G_\alpha^i = \frac{a^{il}}{4} \{ [\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^k} \},$$

where  $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$  and  $(a^{ij}) = (a_{ij})^{-1}$ ,  $F_{x^k} = \frac{\partial F}{\partial x^k}$ ,  $F_{y^k} = \frac{\partial F}{\partial y^k}$ .

For an  $(\alpha, \beta)$ -metric  $L(\alpha, \beta)$  the space  $R^n = (M^n, \alpha)$  is called associated Riemannian space with the Finsler space  $F^n = (M^n, L(\alpha, \beta))$ . The covariant differentiation with respect to the Levi-Civita connection  $\gamma_{jk}^i(x)$  of  $R^n$  is denoted by  $(;)$ . We have the following [3]:

**Lemma 2.1.** *The spray coefficients  $G^i$  are related to  $G_\alpha^i$  by*

$$G^i = G_\alpha^i + \alpha Q s_0^i + J(-2\alpha Q s_0 + r_{00}) \frac{y^i}{\alpha} + H(-2\alpha Q s_0 + r_{00}) \{ b^i - \frac{y^i}{\alpha} \}, \quad (2.2)$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ J &:= \frac{(\phi - s\phi')\phi'}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ H &:= \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')}, \end{aligned}$$

where  $s_{10} = s_{li}y^i$ ,  $s_0 = s_{10}b^l$ ,  $r_{00} = r_{ij}y^i y^j$ ,  $r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i})$ ,  $s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$ ,  $r_j^i = a^{ir}r_{rj}$ ,  $s_j^i = a^{ir}s_{rj}$ ,  $r_j = b_r r_j^r$ ,  $s_j = b_r s_j^r$ ,  $b^i = a^{ir}b_r$  and  $b^2 = a^{rs}b_r b_s$ .

It is well-known that [4] a Finsler metric  $F = F(x, y)$  on an open subset  $U \subset R^n$  is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \quad (2.3)$$

By (2.3), we have the following lemma [13]:

**Lemma 2.2.** *An  $(\alpha, \beta)$ -metric  $F = \alpha\phi(s)$ , where  $s = \beta/\alpha$ , is projectively flat on an open subset  $U \subset R^n$  if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q s_{10} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l \alpha - s y_l) = 0. \quad (2.4)$$

The functions  $G^i(x, y)$  of  $F^n$  with an  $(\alpha, \beta)$ -metric are written in the form [8]

$$2G^i = \gamma_{00}^i + 2B^i, \quad (2.5)$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^* \left\{ \frac{\beta L_\beta}{\alpha L_\alpha} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left( \frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \quad (2.6)$$

provided  $\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$ , where  $\gamma^2 = b^2\alpha^2 - \beta^2$ ,  $L_\alpha = \partial L / \partial \alpha$ ,  $L_\beta = \partial L / \partial \beta$  and  $L_{\alpha\alpha} = \partial^2 L / \partial \alpha^2$ , the subscript 0 means the contraction by  $y^i$  and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}. \quad (2.7)$$

We shall denote the homogeneous polynomials in  $(y^i)$  of degree  $r$  by  $hp(r)$  for brevity. For example,  $\gamma_{00}^i$  is  $hp(2)$ . From (2.5) the Berwald connection  $B\Gamma = (G_{jk}^i, G_j^i, 0)$  of  $F^n$  with an  $(\alpha, \beta)$ -metric is given by

$$\begin{aligned} G_j^i &= \dot{\partial}_j G^i = \gamma_{0j}^i + B_j^i, \\ G_{jk}^i &= \dot{\partial}_k G_j^i = \gamma_{jk}^i + B_{jk}^i, \end{aligned}$$

where we put  $B_j^i = \dot{\partial}_j B^i$  and  $B_{jk}^i = \dot{\partial}_k B_{jk}^i$ . On account of [8],  $B_{jk}^i$  is determined by

$$L_\alpha B_{ji}^t y^j y_t + \alpha L_\beta (B_{ji}^t b_t - b_{ji}) y^j = 0, \quad (2.8)$$

where  $y_k = a_{ik} y^i$ . A Finsler space  $F^n$  with an  $(\alpha, \beta)$ -metric is a Douglas space if and only if  $B^{ij} \equiv B^i y^j - b^j y^i$  is  $hp(3)$  [2]. From (2.6)  $B^{ij}$  is written as follows:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \quad (2.9)$$

### 3. PROJECTIVELY FLAT $(\alpha, \beta)$ -METRIC

In this section, we consider the metric  $F = \frac{\alpha^2}{\alpha - \beta} + \beta$  which is obtained by the Randers change of the Matsumoto metric.

$$F = \alpha \phi(s), \quad \phi(s) = \frac{1 + s - s^2}{1 - s}, \quad s = \frac{\beta}{\alpha}, \quad (3.1)$$

where  $s < 1$  so that  $\phi$  must be a positive function. Let  $b_0$  be the largest number such that

$$(\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_0),$$

that is,

$$\frac{1 - 3s + 2b^2}{1 - s} > 0, \quad (|s| \leq b < b_0).$$

**Lemma 3.1.**  $F = \frac{\alpha^2}{\alpha - \beta} + \beta$  is a Finsler metric if and only if  $\|\beta\|_\alpha < \frac{1}{2}$ .

**Proof.** If  $F = \frac{\alpha^2}{\alpha - \beta} + \beta$  is a Finsler metric, then

$$\frac{1 - 3s + 2b^2}{1 - s} > 0.$$

Let  $s = b$ , we get  $b < \frac{1}{2}$ ,  $\forall b < b_0$ . Let  $b \rightarrow b_0$ , then  $b_0 < \frac{1}{2}$ . So  $\|\beta\|_\alpha < \frac{1}{2}$ . Now, if

$$|s| \leq b < \frac{1}{2},$$

then

$$\frac{1 - 3s + 2b^2}{1 - s} = \frac{1 - 3s + 2s^2 + 2(b^2 - s^2)}{1 - s} > \frac{(2s - 1)(s - 1)}{1 - s} > 0.$$

Thus,  $F = \frac{\alpha^2}{\alpha - \beta} + \beta$  is a Finsler metric.

By Lemma 2.1, the spray coefficients  $G^i$  of  $F$  are given by (2.2) with

$$\begin{aligned} Q &= \frac{2 - 2s + s^2}{1 - 2s} = \frac{2\alpha^2 - 2\alpha\beta + \beta^2}{\alpha(\alpha - 2\beta)}, \\ J &= \frac{(1 - 2s)(2 - 2s + s^2)}{2(1 - 3s + 2b^2)(1 + s - s^2)} = \frac{(\alpha - 2\beta)(2\alpha^2 - 2\alpha\beta + \beta^2)}{2(\alpha - 3\beta + 2\alpha\beta^2)(\alpha^2 + \alpha\beta - \beta^2)}, \\ H &= \frac{1}{1 - 3s + 2b^2} = \frac{\alpha}{\alpha - 3\beta + 2\alpha\beta^2}. \end{aligned}$$

Equation (2.4) is reduced to the following form:

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \frac{\alpha^2(2\alpha^2 - 2\alpha\beta + \beta^2)}{(\alpha - 2\beta)}s_{l0} + \frac{\alpha^2}{(\alpha - 3\beta + 2\alpha b^2)} \left[ \frac{-2(2\alpha^2 - 2\alpha\beta + \beta^2)}{(\alpha - 2\beta)}s_0 + r_{00} \right] (b_l\alpha - \frac{\beta}{\alpha}y_l) = 0. \quad (3.2)$$

We use the following result [15]:

**Lemma 3.2.** *If  $(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = 0$ , then  $\alpha$  is projectively flat.*

By (3.2), we can prove the following:

**Theorem 3.3.** *The  $(\alpha, \beta)$ -metric  $F = \frac{\alpha^2}{\alpha - \beta} + \beta$  is locally projectively flat if and only if*

- (i)  $\beta$  is parallel with respect to  $\alpha$
- (ii)  $\alpha$  is locally projectively flat, i.e., of constant curvature.

**Proof.** Suppose that  $F$  is locally projectively flat. First, we rewrite (3.2) as a polynomial in  $y^i$  and  $\alpha$ . This gives

$$(\alpha - 2\beta)(\alpha - 3\beta + 2\alpha b^2)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^2(2\alpha^2 - 2\alpha\beta + \beta^2)(\alpha - 3\beta + 2\alpha b^2)s_{l0} + \alpha[-2(2\alpha^2 - 2\alpha\beta + \beta^2)s_0 + (\alpha - 2\beta)r_{00}](b_l\alpha^2 - \beta y_l) = 0. \quad (3.3)$$

Because  $\alpha^{even}$  is a polynomial in  $y^i$ , the coefficients of  $\alpha$  must be zero and we obtain

$$\beta(5+4b^2)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = [2\alpha^4(1+2b^2) + \alpha^2\beta^2(7+2b^2)]s_{l0} - 2[(2\alpha^2 + \beta^2)s_0 + \beta r_{00}](b_l\alpha^2 - \beta y_l) \quad (3.4)$$

and

$$[\alpha^2(1+2b^2) + 6\beta^2](a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = [4\alpha^4\beta(2+b^2) + 3\alpha^2\beta^3]s_{l0} - \alpha^2[4\beta s_0 + r_{00}](b_l\alpha^2 - \beta y_l). \quad (3.5)$$

Contracting (3.4) and (3.5) with  $b^l$ , we get

$$\beta(5+4b^2)(b_m\alpha^2 - y_m\beta)G_\alpha^m = [2\alpha^4(1+2b^2) + \alpha^2\beta^2(7+2b^2)]s_0 - 2[(2\alpha^2 + \beta^2)s_0 + \beta r_{00}](b^2\alpha^2 - \beta^2) \quad (3.6)$$

and

$$[\alpha^2(1+2b^2) + 6\beta^2](b_m\alpha^2 - y_m\beta)G_\alpha^m = [4\alpha^4\beta(2+b^2) + 3\alpha^2\beta^3]s_0 - \alpha^2[4\beta s_0 + r_{00}](b^2\alpha^2 - \beta^2). \quad (3.7)$$

(3.6)  $\times \alpha^2$  - (3.7)  $\times 2\beta$  yields

$$3\beta(b_m\alpha^2 - y_m\beta)G_\alpha^m = \alpha^2(2\alpha^2 + 3\beta^2)s_0. \quad (3.8)$$

The polynomial  $\alpha^2(2\alpha^2 + 3\beta^2)$  is not divisible by  $\beta$  and  $\beta$  is not divisible by  $\alpha^2(2\alpha^2 + 3\beta^2)$ . Thus  $s_0$  is divisible by  $\beta$  and  $(b_m\alpha^2 - y_m\beta)G_\alpha^m$  is divisible by  $\alpha^2(2\alpha^2 + 3\beta^2)$ . Therefore, there exist scalar functions  $\tau = \tau(x)$ ,  $\chi = \chi(x)$  such that

$$s_0 = \tau\beta, \quad (3.9)$$

$$(b_m\alpha^2 - y_m\beta)G_\alpha^m = \chi\alpha^2(2\alpha^2 + 3\beta^2). \quad (3.10)$$

Then (3.8) becomes

$$3\beta\chi\alpha^2(2\alpha^2 + 3\beta^2) = \alpha^2(2\alpha^2 + 3\beta^2)\tau\beta.$$

Thus  $\tau = 3\chi$ . Then (3.7) becomes

$$[2\alpha^4(1+2b^2) - 3\alpha^2\beta^2(3-2b^2) - 3\beta^4]\chi = -r_{00}(b^2\alpha^2 - \beta^2). \quad (3.11)$$

Since  $(b^2\alpha^2 - \beta^2)$  is not divisible by  $2\alpha^4(1 + 2b^2) - 3\alpha^2\beta^2(3 - 2b^2) - 3\beta^4$ , it follows from (3.11) that  $\chi = 0$ . By (3.9), (3.10) and (3.11), we get

$$s_0 = 0, \quad (3.12)$$

$$(b_m\alpha^2 - y_m\beta)G_\alpha^m = 0, \quad (3.13)$$

$$\text{and } r_{00} = 0. \quad (3.14)$$

Then substituting (3.12) and (3.13) into (3.5), we get

$$s_{i0} = 0. \quad (3.15)$$

Then by (3.13) and Lemma 3.2,  $\alpha$  is projectively flat. And by (3.14) and (3.15),  $b_{i;j} = 0$ . i.e.,  $\beta$  is parallel with respect to  $\alpha$ .

Conversely, if  $\beta$  is parallel with respect to  $\alpha$  and  $\alpha$  is locally projectively flat, then by Lemma 2.2, we can easily see that  $F$  is locally projectively flat.

#### 4. BERWALD AND DOUGLAS SPACES

In this section, we find the condition that a Finsler space  $F^n$  with  $(\alpha, \beta)$ -metric (3.1) be a Berwald space. In the  $n$ -dimensional Finsler space  $F^n$  with an  $(\alpha, \beta)$ -metric (3.1), we have

$$\begin{aligned} L_\alpha &= \frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}, & L_\beta &= \frac{2\alpha(\alpha-\beta)+\beta^2}{(\alpha-\beta)^2}, \\ L_{\alpha\alpha} &= \frac{2\beta^2}{(\alpha-\beta)^3}, & L_{\beta\beta} &= \frac{2\alpha^2}{(\alpha-\beta)^3}. \end{aligned} \quad (4.1)$$

Substituting (4.1) into (2.8), we have

$$\alpha [B_{ji}^t y^j y_t + 2\alpha(B_{ji}^t b_t - b_{j;i})y^j] + \beta [\beta(B_{ji}^t b_t - b_{j;i})y^j - 2 \{B_{ji}^t y^j y_t + 2\alpha(B_{ji}^t b_t - b_{j;i})y^j\}] = 0. \quad (4.2)$$

Assume that  $F^n$  is a Berwald space, that is,  $G_{jk}^i = G_{jk}^i(x)$ . Then we have  $B_{ji}^t = B_{ji}^t(x)$ . since  $\alpha$  is irrational in  $(y^i)$ , from (4.2), we have

$$B_{ji}^t y^j y_t + 2\alpha(B_{ji}^t b_t - b_{j;i})y^j = 0, \quad (4.3)$$

$$\beta(B_{ji}^t b_t - b_{j;i})y^j - 2 \{B_{ji}^t y^j y_t + 2\alpha(B_{ji}^t b_t - b_{j;i})y^j\} = 0. \quad (4.4)$$

From (4.3) and (4.4), we obtain

$$B_{ji}^t y^j y_t = 0 \quad \text{and} \quad (B_{ji}^t b_t - b_{j;i})y^j = 0,$$

which show

$$B_{ji}^t a_{th} + B_{hi}^t a_{tj} = 0 \quad \text{and} \quad B_{ji}^t b_t - b_{j;i} = 0.$$

Thus by the well known Christoffel process we get  $B_{ji}^t = 0$ . Therefore we have

**Theorem 4.1.** *Randers change of the Matsumoto metric 3.1) is a Berwald metric if and only if  $b_{j;i} = 0$ , and then the Berwald connection is Riemannian  $(\gamma_{jk}^i, \gamma_{0j}^i, 0)$ .*

Now, we consider the condition that a Finsler space  $F^n$  with an  $(\alpha, \beta)$ -metric (3.1) be a Douglas space.

Substituting (4.1) into (2.9), we obtain

$$\begin{aligned} (\alpha - 2\beta)\{\alpha^2 - 3\alpha\beta + 2\beta^2 + 2\gamma^2\}B^{ij} + \{8\alpha^3\beta - 2\alpha^4 - 11\alpha^2\beta^2 - 4\gamma^2\alpha^2 + 7\alpha\beta^3 + 4\alpha\beta\gamma^2 - 2\beta^4 - 2\beta^2\gamma^2\} \\ (s_0^i y^j - s_0^j y^i) + \{(2\alpha^2\beta - \alpha^3)r_{00} + (4\alpha^4 - 4\alpha^3\beta + 2\alpha^2\beta^2)s_0\}(b^i y^j - b^j y^i) = 0. \end{aligned} \quad (4.5)$$

We note that  $\beta^2 L_\alpha + \gamma^2 L_{\beta\beta} \neq 0$ .

Suppose that  $F^n$  is a Douglas space, that is  $B^{ij}$  are  $hp(3)$ . Separating (4.5) in rational and irrational terms of  $y^i$ , because  $\alpha$  is irrational in  $(y^i)$ , we have

$$\begin{aligned} & \left[ \{4\beta^3 + 4\beta\gamma^2 + 5\alpha^2\beta\}B^{ij} + \{2\beta^4 + 2\beta^2\gamma^2 + \alpha^2(11\beta^2 + 4\gamma^2) + 2\alpha^2\}(s_0^i y^j - s_0^j y^i) \right. \\ & \quad - \{\alpha^2(2\beta r_{00} + 2\beta^2 s_0) + 4\alpha^4 s_0\}(b^i y^j - b^j y^i) \left. \right] + \alpha \left[ -(\alpha^2 + 8\beta^2 + 2\gamma^2)B^{ij} \right. \\ & \quad \left. - (8\alpha^2\beta + 7\beta^3 + 4\beta\gamma^2)(s_0^i y^j - s_0^j y^i) + (\alpha^2 r_{00} + 4\alpha^2)(b^i y^j - b^j y^i) \right] = 0. \end{aligned} \quad (4.6)$$

Hence the equation (4.6) is divided into two equations as follows:

$$\begin{aligned} & \{4\beta^3 + 4\beta\gamma^2 + 5\alpha^2\beta\}B^{ij} + \{2\beta^4 + 2\beta^2\gamma^2 + \alpha^2(11\beta^2 + 4\gamma^2) + 2\alpha^2\}(s_0^i y^j - s_0^j y^i) \\ & \quad - \{\alpha^2(2\beta r_{00} + 2\beta^2 s_0) + 4\alpha^4 s_0\}(b^i y^j - b^j y^i) = 0 \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} & -(\alpha^2 + 8\beta^2 + 2\gamma^2)B^{ij} - (8\alpha^2\beta + 7\beta^3 + 4\beta\gamma^2)(s_0^i y^j - s_0^j y^i) \\ & \quad + (\alpha^2 r_{00} + 4\alpha^2)(b^i y^j - b^j y^i) = 0. \end{aligned} \quad (4.8)$$

Eliminating  $B^{ij}$  from (4.7) and (4.8), we obtain

$$P(s_0^i y^j - s_0^j y^i) + \alpha^2 Q(b^i y^j - b^j y^i) = 0, \quad (4.9)$$

where

$$P = 23\alpha^2\beta^4 + 4\alpha^2\beta^2\gamma^2 - 13\alpha^4\beta^2 + 2\alpha^6 + 8\alpha^4\gamma^2 - 12\beta^6 - 24\beta^4\gamma^2 - 12\gamma^4\beta^2 + 8\gamma^4\alpha^2, \quad (4.10)$$

$$Q = \alpha^2\beta r_{00} - 14\alpha^2 s_0\beta^2 - 4\alpha^4 s_0 - 12\beta^3 r_{00} + 12\beta^2\gamma^2 s_0 - 8\alpha^2\gamma^2 s_0 = 0. \quad (4.11)$$

Transvection of (4.9) by  $b_i y_j$ , we get

$$P s_0 + Q \gamma^2 = 0. \quad (4.12)$$

The term of (4.12) which does not contain  $\alpha^2$  is  $12\beta^5(\beta s_0 + r_{00})$ . Hence there exists  $hp(5)$ :  $V_5$  such that

$$12\beta^5(\beta s_0 + r_{00}) = \alpha^2 V_5. \quad (4.13)$$

Here we consider two cases.

(i)  $V_5 = 0$ , (ii)  $V_5 \neq 0$ ,  $\alpha^2 \not\equiv 0 \pmod{\beta}$ . Case (i): This leads to  $r_{00} = -\beta s_0$ , i.e.,  $2r_{ij} = -\{b_i s_j + b_j s_i\}$ . Therefore, substituting  $r_{00} = -\beta s_0$  into (4.12), we get

$$s_0(P + \gamma^2 Q_1) = 0, \quad (4.14)$$

where

$$Q_1 = 12\beta^4 + 12\beta^2\gamma^2 - 3\alpha^2\beta^2 - 14\alpha^2\beta^2 - 8\alpha^2\gamma^2.$$

If  $P + \gamma^2 Q_1 = 0$ . The term of  $P + \gamma^2 Q_1 = 0$  which does not contain  $\alpha^2$  is  $12\beta^4(3 - b^2)$ . Thus there exists  $hp(2)$ :  $V_2$  such that

$$12\beta^4(3 - b^2) = \alpha^2 V_2,$$

where we assume  $b^2 \neq 3$ . Hence we have  $V_2 = 0$ , which leads to a contradiction, that is,  $P + \gamma^2 Q_1 \neq 0$ . Therefore, we have  $s_0 = 0$  from (4.14) and we obtain  $r_{00} = 0$ . Substituting  $s_0 = 0$  and  $r_{00} = 0$  into (4.9), we get

$$P(s_0^i y^j - s_0^j y^i) = 0. \quad (4.15)$$

If  $P = 0$ , then from (4.10), we have

$$P = 23\alpha^2\beta^4 + 4\alpha^2\beta^2\gamma^2 - 13\alpha^4\beta^2 + 2\alpha^6 + 8\alpha^4\gamma^2 - 12\beta^6 - 24\beta^4\gamma^2 - 12\gamma^4\beta^2 + 8\gamma^4\alpha^2 = 0. \quad (4.16)$$

The term of (4.16) which does not contain  $\alpha^2$  is  $27\beta^4$ . Thus there exists  $hp(2)$ :  $V_2$  such that

$$27\beta^4 = \alpha^2 V_2,$$

from which we have  $V_2 = 0$ . It is a contradiction, that is  $P \neq 0$ . Therefore, from (4.15) we obtain

$$s_0^i y^j - s_0^j y^i = 0.$$

Transvecting the above equation by  $y_j$  gives  $s_0^i = 0$ , which imply  $s_{ij} = 0$ . Consequently, we have  $r_{ij} = s_{ij} = 0$ , that is,  $b_{i;j} = 0$ .

Case (ii): The equation (4.13) shows that there exists a function  $k = k(x)$  satisfying

$$\beta s_0 + r_{00} = k(x)\alpha^2. \quad (4.17)$$

Thus we have the term of (4.12) does not contain  $\alpha^2$  is included in the term:  $12\beta^4\{(3 - b^2)s_0 + k\beta\}$ . Thus there exists  $hp(3)$ :  $V_3$  such that

$$12\beta^4\{(3 - b^2)s_0 + k\beta\} = \alpha^2 V_3.$$

From  $\alpha^2 \not\equiv 0 \pmod{\beta}$ , it follows that  $V_3$  must vanish and hence we have

$$s_0 = -\frac{k(x)}{3 - b^2}\beta. \quad (4.18)$$

From (4.18), we have  $s_i = -k(x)b_i/(3 - b^2)$ . Transvecting (4.18) by  $b^i$  leads to  $k(x)b^2 = 0$ . Hence we get  $k(x) = 0$ . Substituting  $k(x) = 0$  into (4.17) and (4.18), we obtain  $s_0 = 0$  and  $r_{00} = 0$ . From (4.15), we have  $P(s_0^i y^j - s_0^j y^i) = 0$ . If  $P = 0$ , then it is a contradiction. Hence  $P \neq 0$ . Therefore, we obtain  $s_0^i y^j - s_0^j y^i = 0$ . Transvecting this equation by  $y_j$  we get  $s_0^i = 0$ .

Hence both the cases (i) and (ii) lead to  $r_{ij} = 0$  and  $s_{ij} = 0$ , that is,  $b_{i;j} = 0$ . Conversely if  $b_{i;j} = 0$ , then  $F^n$  is a Berwald space, so  $F^n$  is a Douglas space.

Thus we have the following

**Theorem 4.2.** *Randers change of Matsumoto metric with  $b^2 \neq 3$  is of Douglas type if and only if  $\alpha^2 \not\equiv 0 \pmod{\beta}$  and  $b_{i;j}=0$ .*

From Theorem 4.1 and Theorem 4.2, we have

**Theorem 4.3.** *If Randers change of Matsumoto metric with  $b^2 \neq 3$  is of Douglas type, then it is Berwaldian.*

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