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DISTANCE-BASED ENERGIES OF NON-COMMUTING GRAPH FOR DIHEDRAL GROUPS

MAMIKA UJIANITA ROMDHINI*, ATHIRAH NAWAWI, FAISAL AL-SHARQI ABDURAHIM, ANDIKA ELLENA SAUFIKA HAKIM MAHARANI, IFAN HASNAN DANI

Abstract. The connection of graph and group is a new interesting topic in which the group elements are the graph's vertex set. The non-commuting graph for dihedral groups is one graph representation of the group. This research focuses on the spectral properties of this graph, including spectral radius and energy, in which the matrix representations are distance, Harary, distance Laplacian, and distance signless Laplacian matrices. The relationships between the obtained energies are also presented in this paper.

1. Introduction

The link between graphs and matrices, or more precisely, how algebra facilitates the study of graphs, is the focus of spectral graph theory. Lately, the study of graph spectral has extended to the graph defined on a finite group. The best example of this graph is the non-commuting graph.

Let G be a finite group and $Z(G)$ is the center of G. The non-commuting graph of G is denoted by $\Omega_G = (V(\Omega_G), E(\Omega_G))$ with $V(\Omega_G) = G\backslash Z(G)$ as the set of vertex and $E(\Omega_G)$ is the set of edge. The element of $E(\Omega_G)$ is (u, v) where $uv \neq vu$, for all $u, v \in V(\Omega_G)$ [1]. Abdollahi [1] studied some properties of Γ_G associated with a non-abelian group G, and proved that Γ_G is always connected. As a result, the notion of distance between any two vertices in Γ_G is well defined. In this work, we concentrate on the dihedral group of order $2n$, D_{2n}, with $n \geq 3$. Throughout the paper, the non-commuting graph of D_{2n} will be denoted by $\Omega_{D_{2n}}$.

Gutman [10] pioneered the energy of Ω_G in 1978. It is known that the energy of a graph can not take an odd integer value [5, 18]. Moreover, bounds for the adjacency energy have been established in the literature, see [9]. In 2008, Indulal and Gutman [14] proposed a distance-based matrix defined in terms of the pairwise vertex distances of a graph, and subsequently they introduced the concept of distance energy.

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The distance-based energies of the graphs have been discussed and are defined in terms of several matrices, including Wiener-Hosoya [13], distance [14], and distance signless Laplacian matrices [19]. Another type of matrix based on the distance defined by [6], is a resistance-distance matrix. Aouchiche and Hansen [4] were the first to introduce distance Laplacian and distance signless Laplacian matrices, then in 2024, Jahanbani et al. [16] extended the study to the eigenvalues of this matrix. Ivanciuc et al. [15] introduce a novel molecular graph matrix, the Harary matrix (originally termed the reciprocal distance matrix). Meanwhile, the Wiener-Hosoya energy of Ω_G has been done by [22], now we focus on distance, Harary, distance Laplacian, and distance signless Laplacian matrices in this work.

2. Preliminaries

In this research, we continue to discuss $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$, where $n \geq 3$ and divide it into two sets $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n}) \subset D_{2n}$, and $G_2 = \{a^i b : 1 \leq i \leq n\} \subset D_{2n}$. consider $u, v \in V(\Omega_{D_{2n}})$, their distance is represented by d_{pq}. Then we provide our previous result from [21] on d_{pq} in $\Omega_{D_{2n}}$.

Theorem 2.1. [21] *In* $\Omega_{D_{2n}}$*,*

(1) *for odd* n, $d_{uv} = \begin{cases} 2, & \text{if } u, v \in G_1 \\ 1, & \text{otherwise,} \end{cases}$,

(2) *for even* n, $d_{uv} = \begin{cases} 2, & \text{if } u, v \in G_1 \\ 2, & u \in G_2, v \in \left\{a^{\frac{n}{2}+i}b\right\} \\ 1, & \text{otherwise.} \end{cases}$

Furthermore, for $u \in V(\Omega_{D_{2n}})$, The quantity τ_u represents the transmission of u and is obtained by summing d_{uv}, for all $v \in V(\Omega_{D_{2n}})$. The following theorem is our previous result [22] which is useful for matrix construction in the next section.

Theorem 2.2. [22] *In* $\Omega_{D_{2n}}$*, the transmission of vertex* u *is*

(1) *for odd* n, $\tau_u = \begin{cases} 3n-4, & \text{if } u \in G_1 \\ 2(n-1), & \text{if } u \in G_2, \end{cases}$ *, and*

(2) *for even* n, $\tau_u = \begin{cases} 3(n-2), & \text{if } u \in G_1 \\ 2(n-1), & \text{if } u \in G_2, \end{cases}$

The transmission-based matrices of $\Omega_{D_{2n}}$ are based on the following definitions.

Definition 1. [13] *Transmission* T *of* $\Omega_{D_{2n}}$ *is* $T(\Omega_{D_{2n}}) = diag(\tau_{u_1}, \tau_{u_2}, \ldots, \tau_{u_n})$, *where* $u_1, u_2, ..., u_n \in V(\Omega_{D_{2n}})$.

Definition 2. [14] *The distance matrix of* $\Omega_{D_{2n}}$, $D(\Omega_{D_{2n}})$, *is a square matrix with components* d_{uv} *for* $u \neq v$, *and zero if* $u = v$.

Definition 3. [15] *The harary matrix of* $\Omega_{D_{2n}}$, $H(\Omega_{D_{2n}})$, *is represented by a square matrix having entries* $\frac{1}{d_{uv}}$ *for* $u \neq v$, *and zero if* $u = v$.

Definition 4. [4] *The distance Laplacian* (DL) *matrix of* $\Omega_{D_{2n}}$ *is given by*

$$DL(\Omega_{D_{2n}}) = T(\Omega_{D_{2n}}) - D(\Omega_{D_{2n}}).$$

Definition 5. [4] *The distance signless Laplacian* (DSL) *matrix of* $\Omega_{D_{2n}}$ *is given by*

$$DSL(\Omega_{D_{2n}}) = T(\Omega_{D_{2n}}) + D(\Omega_{D_{2n}}).$$

Suppose that the eigenvalues of $D(\Omega_{D_{2n}})$ are $\mu_1, \mu_2, \ldots, \mu_n$, then the spectrum of $\Omega_{D_{2n}}$ can be written as $Spec_D(\Omega_{D_{2n}}) = \left\{\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_n^{k_n}\right\}$, where $k_1, k_2, \ldots, k_n$ are the respective multiplicities.

Definition 6. [10] *The distance energy of* $\Omega_{D_{2n}}$ *is*

$$\varepsilon_D(\Omega_{D_{2n}}) = \sum_{i=1}^{n} |\mu_i| .$$

Definition 7. [11] *The distance-spectral radius of* $\Omega_{D_{2n}}$ *is*

$$\rho_D(\Omega_{D_{2n}}) = max\{|\lambda| : \lambda \in Spec_D(\Omega_{D_{2n}})\}.$$

Definition 8. [17] *Graph* $\Omega_{D_{2n}}$ *is said to be hyperenergetic whenever its* $D-$*energy meets the subsequent requirements.*

$$\varepsilon_D(\Omega_{D_{2n}}) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1)-2, & \text{for even } n, \end{cases}$$

since for odd n, Γ_G *consists of* $2n-1$ *vertices, and for even* n, *it is* $2n-2$ *vertices.*

The preceding notations and definitions remain valid for Harary, distance Laplacian, and distance signless Laplacian matrices.

Theorem 2.3. [20] *Consider four real numbers denoted by Let* $a, b, c,$ *and* d. *The characteristic equation of a* $(2n-1) \times (2n-1)$ *matrix,*

$$M = \begin{pmatrix} (a+d)I_{n-1} - dJ_{n-1} & -bJ_{n-1\times n} \\ -bJ_{n\times n-1} & (c+b)I_n - bJ_n \end{pmatrix}$$

can be simplified into an expression as

$$P_M(\mu) = (\mu-a-d)^{n-2}(\mu-b-c)^{n-1}\left((\mu-c+b(n-1))(\mu-a+(n-2)d)+(1-n)nb^2\right).$$

Theorem 2.4. [23] *Let* a, b, c *and* d *be real-valued parameters. The characteristic equation of a* $(2n-2) \times (2n-2)$ *matrix*

$$M = \begin{bmatrix} (a+d)\,I_{n-2} - dJ_{n-2} & -cJ_{(n-2)\times\frac{n}{2}} & -cJ_{(n-2)\times\frac{n}{2}} \\ -cJ_{\frac{n}{2}\times(n-2)} & (b+c)\,I_{\frac{n}{2}} - cJ_{\frac{n}{2}} & (-d+c)I_{\frac{n}{2}} - cJ_{\frac{n}{2}} \\ -cJ_{\frac{n}{2}\times(n-2)} & (-d+c)I_{\frac{n}{2}} - cJ_{\frac{n}{2}} & (b+c)\,I_{\frac{n}{2}} - cJ_{\frac{n}{2}} \end{bmatrix}$$

is

$$\begin{aligned} P_M(\lambda) =& (\mu-a-d)^{n-3}(\mu+d-b-2c)^{\frac{n}{2}-1}(\mu-b-d)^{\frac{n}{2}} \\ & \left(\mu^2+((n-2)(d+c)-a-b)\mu+(d(n-3)-a)(d-b+(n-2)c)-n(n-2)c^2\right). \end{aligned}$$

3. Main Results

3.1. Distance Energy.

Theorem 3.1. *In* $\Omega_{D_{2n}}$, *the spectrum of* $D(\Omega_{D_{2n}})$ *is*

(1) *for* n *is odd:*

$$Spec_D(\Omega_{D_{2n}}) = \left\{ \left(\frac{3n-5}{2} + \frac{\sqrt{5n^2-10n+9}}{2}\right)^1, \left(\frac{3n-5}{2} - \frac{\sqrt{5n^2-10n+9}}{2}\right)^1, (-1)^{n-1}, (-2)^{n-2} \right\}.$$

(2) *for n is even:*

$$Spec_D(\Omega_{D_{2n}}) = \left\{ \left(\frac{3n-2}{2} + \frac{\sqrt{5n^2-36n+40}}{2}\right)^1, \left(\frac{3n-2}{2} - \frac{\sqrt{5n^2-36n+40}}{2}\right)^1, 0^{\frac{n}{2}-1}, (-2)^{n-3+\frac{n}{2}} \right\}.$$

Proof. (1) We have odd n, by Theorem 2.1, it is easy to see that the distance between each pair of $a, a^2, \ldots, a^{n-1}$ is 2, and it is equal to 1, otherwise. According to Definition 2, then the entries of $D(\Omega_{D_{2n}}) = [m_{pq}]$ of size $(2n-1) \times (2n-1)$ are

(a) $m_{pq} = 2$ for $p, q = 1, 2, \ldots, n-1$ and $p \neq q$;
(b) $m_{pq} = 1$ for $p = 1, 2, \ldots, n-1$ and $q = n, n+1, \ldots, 2n-1$ or conversely;
(c) $m_{pq} = 1$ for $p, q = n, n+1, \ldots, 2n-1$ and $p \neq q$;
(d) $m_{pq} = 0$ for $p = q$.

Therefore,

$$D(\Omega_{D_{2n}}) = \begin{array}{c} a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \overset{\begin{array}{cccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{n-1}b \end{array}}{\begin{pmatrix} 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 \\ 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}}.$$

It is the form of block matrices as follows

$$D(\Omega_{D_{2n}}) = \begin{pmatrix} 2(J-I)_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix}.$$

The above determinant is similar to the form of Theorem 2.3. By choosing $a = 0$, $b = -1$, $c = 0$, $d = -2$ and $n_1 = n-1$, $n_2 = n$, then we derive the characteristic polynomial of $D(\Omega_{D_{2n}})$ as given below:

$$P_{D(\Omega_{D_{2n}})}(\mu) = (\mu+2)^{n-2} (\mu+1)^{n-1} \left(\mu^2 - (3n-5)\mu + (n-1)(n-4)\right). \quad (3.1)$$

According to Equation 3.1, we obtain four eigenvalues which are $\mu_1 = -2$ of multiplicity $(n-2)$, $\mu_2 = -1$ of multiplicity $(n-1)$, and $\mu_{3,4} = \frac{3n-5}{2} \pm \frac{\sqrt{5n^2-10n+9}}{2}$. Thus, the spectrum of $D(\Omega_{D_{2n}})$ is

$$Spec_D(\Omega_{D_{2n}}) = \left\{ \left(\frac{3n-5}{2} + \frac{\sqrt{5n^2-10n+9}}{2}\right)^1, \left(\frac{3n-5}{2} - \frac{\sqrt{5n^2-10n+9}}{2}\right)^1, (-1)^{n-1}, (-2)^{n-2} \right\}.$$

(2) Suppose that n is even. By Theorem 2.1 we have the distance between $a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}$ is always equal to 2 and the distance between $a^i b$ and $a^{\frac{n}{2}+i} b$ is also 2 for $i = 1, 2, \ldots, n$. Otherwise, the distance is 1. Based on Definition 2, then the entries of $D(\Omega_{D_{2n}})$ of size $(2n-2) \times (2n-2)$ are

(a) $m_{pq} = 2$ for $p, q = 1, 2, \ldots, n-2$ and $p \neq q$;
(b) $m_{pq} = 1$ for $p = 1, 2, \ldots, n-2$ and $q = n-1, n, n+1, \ldots, 2n-2$ or conversely;
(c) $m_{pq} = 2$ for $p = n-2+i$ and $q = n-2+\frac{n}{2}+i$ or vice versa for $i = 1, 2, \ldots, \frac{n}{2}$;
(d) $m_{pq} = 1$ for $p, q = n-1, n, n+1, \ldots, 2n-2$ excluding ($p = n-2+i$ and $q = n-2+\frac{n}{2}+i$ for $i = 1, 2, \ldots, \frac{n}{2}$) or conversely, and $p \neq q$;
(e) $m_{pq} = 0$ for $p = q$.

This implies

$$D(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ a^{\frac{n}{2}+1}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 1 & 1 & 2 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 0 & 1 & 1 & \ldots & 2 \\ 1 & 1 & \ldots & 1 & 2 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 1 & 2 & \ldots & 1 & 1 & 0 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 2 & 1 & 1 & \ldots & 0 \end{pmatrix} \end{array}.$$

It is a nine-block matrix as given below:

$$D(\Omega_{D_{2n}}) = \begin{bmatrix} 2(J-I)_{n-2} & J_{(n-2)\times\frac{n}{2}} & J_{(n-2)\times\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J-I)_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} + 2I_{\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J-I)_{\frac{n}{2}} + 2I_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} \end{bmatrix}.$$

The matrix $D(\Omega_{D_{2n}})$ is similar form with Theorem 2.4 where $a = b = 0$, $c = -1$, $d = -2$, then

$$P_{D(\Omega_{D_{2n}})}(\mu) = \mu^{\frac{n}{2}-1}(\mu+2)^{n-3+\frac{n}{2}}\left(\mu^2 - 3(n-2)\mu + n(n-4)\right).$$

Based on Theorem $P_{D(\Omega_{D_{2n}})}(\mu)$ above, the eigenvalues of $D(\Omega_{D_{2n}})$ are $\mu_1 = -2$ of multiplicity $n-3+\frac{n}{2}$, $\mu_2 = 0$ of multiplicity $\frac{n}{2}-1$ and $\mu_{3,4} = \frac{3n-2}{2} \pm \frac{\sqrt{5n^2-36n+40}}{2}$. Consequently, we can write

$$Spec_D(\Omega_{D_{2n}}) = \left\{ \left(\frac{3n-2}{2} + \frac{\sqrt{5n^2-36n+40}}{2}\right)^1, \left(\frac{3n-2}{2} - \frac{\sqrt{5n^2-36n+40}}{2}\right)^1, 0^{\frac{n}{2}-1}, (-2)^{n-3+\frac{n}{2}} \right\},$$

and this proof is now complete.

□

After the spectrum of $D(\Gamma_G)$ was constructed, the spectral radius and energy can be proved as follows:

Theorem 3.2. *In $\Omega_{D_{2n}}$, the distance spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_D(\Omega_{D_{2n}}) = \begin{cases} \frac{3n-5}{2} - \frac{\sqrt{5n^2-10n+9}}{2}, & \text{if } n \text{ is odd} \\ \frac{3n-2}{2} + \frac{\sqrt{5n^2-36n+40}}{2}, & \text{if } n \text{ is even} \end{cases}.$$

Proof. In view of Theorem 3.1 together with Definition 7, we obtain the spectral radius of $D(\Omega_{D_{2n}})$ as the maximum of μ_i, for $i = 1, 2, 3, 4$ and this concludes the argument. □

Theorem 3.3. *In $\Omega_{D_{2n}}$, the distance energy of $\Omega_{D_{2n}}$ is*

(1) *for n is odd:* $\varepsilon_D(\Omega_{D_{2n}}) = \begin{cases} 4 + 2\sqrt{6}, & \text{if } n = 3 \\ 2(3n-5), & \text{if } n > 3 \end{cases}$

(2) *for n is even:* $\varepsilon_D(\Omega_{D_{2n}}) = 6(n-2)$.

Proof. (1) Based on Definition 6 and Theorem 3.1 (1) for odd n and $n = 3$, we have two different eigenvalues. Then $\varepsilon_D(\Omega_{D_{2n}}) = 4 + 2\sqrt{6}$. For the $n > 3$ case, the distance energy of $\Omega_{D_{2n}}$ is computed as given below:

$$\varepsilon_D(\Omega_{D_{2n}}) = (n-2)\,|-2| + (n-1)\,|-1| + \left| \frac{3n-5}{2} \pm \frac{\sqrt{5n^2-10n+9}}{2} \right| = 2(3n-5).$$

(2) Suppose that n is even and according to Definition 6 and Theorem 3.1 (2), we get the distance energy of $\Omega_{D_{2n}}$:

$$\begin{aligned} \varepsilon_D(\Omega_{D_{2n}}) &= \left(\frac{n}{2} - 1\right)|0| + \left(n - 3 + \frac{n}{2}\right)|-2| + \left| \frac{3n-2}{2} \pm \frac{\sqrt{5n^2-36n+40}}{2} \right| \\ &= 6(n-2). \end{aligned}$$

□

3.2. **Harary Matrix.** This section is devoted to study the Harary matrix of $\Omega_{D_{2n}}$ in order to find the spectrum and energy of $\Omega_{D_{2n}}$.

Theorem 3.4. *In $\Omega_{D_{2n}}$, the spectrum of $H(\Omega_{D_{2n}})$ is*

(1) *for n is odd:*

$$\begin{aligned} Spec_H(\Omega_{D_{2n}}) = \Bigg\{ &\left(\frac{1}{4}\left(3n - 4 + \sqrt{n(17n-16)}\right)\right)^1, \left(\frac{1}{4}\left(3n - 4 - \sqrt{n(17n-16)}\right)\right)^1, \\ &(-1)^{n-1}, \left(-\frac{1}{2}\right)^{n-2} \Bigg\}. \end{aligned}$$

(2) *for n is even:*

$$\begin{aligned} Spec_H(\Omega_{D_{2n}}) = \Bigg\{ &\left(\frac{1}{4}\left(3(n-2) + \sqrt{n(17n-32)}\right)\right)^1, \left(-\frac{3}{2}\right)^{\frac{n}{2}-1}, \left(-\frac{1}{2}\right)^{n-3+\frac{n}{2}} \\ &\left(\frac{1}{4}\left(3(n-2) - \sqrt{n(17n-32)}\right)\right)^1 \Bigg\}, \end{aligned}$$

Proof. (1) Similarly with the proof of Theorem 3.1 for odd n and by Theorem 2.1 and Definition 3, then the entries of $H(\Omega_{D_{2n}}) = [h_{pq}]$ of size $(2n-1) \times (2n-1)$ are

(a) $h_{pq} = \frac{1}{2}$ for $p, q = 1, 2, \ldots, n-1$ and $p \neq q$;

(b) $h_{pq} = 1$ for $p = 1, 2, \ldots, n-1$ and $q = n, n+1, \ldots, 2n-1$ or conversely;
(c) $h_{pq} = 1$ for $p, q = n, n+1, \ldots, 2n-1$ and $p \neq q$;
(d) $h_{pq} = 0$ for $p = q$.

Therefore,

$$H(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \ldots & \frac{1}{2} & 1 & 1 & \ldots & 1 \\ \frac{1}{2} & 0 & \ldots & \frac{1}{2} & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 0 \end{pmatrix} \end{array}.$$

It is the form of block matrices as follows

$$H(\Omega_{D_{2n}}) = \begin{pmatrix} \frac{1}{2}(J-I)_{n-1} & J_{(n-1)\times n} \\ J_{(n-1)\times n} & (J-I)_n \end{pmatrix}.$$

The above determinant is similar to the form of Theorem 2.3. By choosing $a = 0$, $b = -1$, $c = 0$, $d = -\frac{1}{2}$ and $n_1 = n-1$, $n_2 = n$, then we derive the characteristic polynomial of $H(\Omega_{D_{2n}})$ as given below:

$$P_{H(\Omega_{D_{2n}})}(\mu) = \left(\mu + \frac{1}{2}\right)^{n-2} (\mu+1)^{n-1} \left(\mu^2 - \frac{1}{2}(3n-4)\mu - \frac{1}{2}(n-1)(n+2)\right). \tag{3.2}$$

According to $P_{H(\Omega_{D_{2n}})}(\mu)$, we obtain four eigenvalues which are $\mu_1 = -\frac{1}{2}$ of multiplicity $(n-2)$, $\mu_2 = -1$ having multiplicity $(n-1)$, and $\mu_{3,4} = \frac{1}{4}(3n - 4 \pm \sqrt{n(17n-16)}$. Thus, the spectrum of $H(\Omega_{D_{2n}})$ is

$$Spec_H(\Omega_{D_{2n}}) = \left\{ \left(\frac{1}{4}\left(3n-4+\sqrt{n(17n-16)}\right)\right)^1, \left(\frac{1}{4}\left(3n-4-\sqrt{n(17n-16)}\right)\right)^1, (-1)^{n-1}, \left(-\frac{1}{2}\right)^{n-2} \right\}.$$

(2) For the even n case. By the same argument of Theorem 3.4, the entries of $H(\Omega_{D_{2n}}) = [h_{pq}]$ of size $(2n-2) \times (2n-2)$ are
(a) $h_{pq} = \frac{1}{2}$ for $p, q = 1, 2, \ldots, n-2$ and $p \neq q$;
(b) $h_{pq} = 1$ for $p = 1, 2, \ldots, n-2$ and $q = n-1, n, n+1, \ldots, 2n-2$ or conversely;
(c) $h_{pq} = \frac{1}{2}$ for $p = n-2+i$ and $q = n-2+\frac{n}{2}+i$ or conversely for $i = 1, 2, \ldots, \frac{n}{2}$;
(d) $h_{pq} = 1$ for $p, q = n-1, n, n+1, \ldots, 2n-2$ excluding ($p = n-2+i$ and $q = n-2+\frac{n}{2}+i$ for $i = 1, 2, \ldots, \frac{n}{2}$) or vice versa, and $p \neq q$;
(e) $h_{pq} = 0$ for $p = q$.

This implies

$$H(\Omega_{D_{2n}}) = \begin{array}{c|ccccccccccccc}
 & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \dots & a^{n-1}b \\
\hline
a & 0 & \frac{1}{2} & \dots & \frac{1}{2} & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
a^2 & \frac{1}{2} & 0 & \dots & \frac{1}{2} & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1} & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
b & 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 & \frac{1}{2} & 1 & \dots & 1 \\
ab & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 1 & 1 & \frac{1}{2} & \dots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{\frac{n}{2}-1}b & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 & 1 & 1 & \dots & \frac{1}{2} \\
a^{\frac{n}{2}}b & 1 & 1 & \dots & 1 & \frac{1}{2} & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\
a^{\frac{n}{2}+1}b & 1 & 1 & \dots & 1 & 1 & \frac{1}{2} & \dots & 1 & 1 & 0 & \dots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1}b & 1 & 1 & \dots & 1 & 1 & 1 & \dots & \frac{1}{2} & 1 & 1 & \dots & 0
\end{array}.$$

It is a nine-block matrix as given below:

$$H(\Omega_{D_{2n}}) = \begin{bmatrix} \frac{1}{2}(J-I)_{n-2} & J_{(n-2)\times\frac{n}{2}} & J_{(n-2)\times\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J-I)_{\frac{n}{2}} & \frac{1}{2}(J-I)_{\frac{n}{2}} + \frac{1}{2}I_{\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & \frac{1}{2}(J-I)_{\frac{n}{2}} + \frac{1}{2}I_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} \end{bmatrix}.$$

The matrix $H(\Omega_{D_{2n}})$ is similar form with Theorem 2.4 where $a = b = 0$, $c = -1$, $d = -\frac{1}{2}$, then

$$P_{H(\Omega_{D_{2n}})}(\mu) = \left(\mu + \frac{3}{2}\right)^{\frac{n}{2}-1} \left(\mu + \frac{1}{2}\right)^{n-3+\frac{n}{2}} \left(\mu^2 - \frac{3}{2}(n-2)\mu - \frac{1}{4}(2n^2+n-9)\right).$$

Based on Theorem $P_{H(\Omega_{D_{2n}})}(\mu)$ above, the eigenvalues of $H(\Omega_{D_{2n}})$ are $\mu_1 = -\frac{1}{2}$ of multiplicity $n-3+\frac{n}{2}$, $\mu_2 = -\frac{3}{2}$ of multiplicity $\frac{n}{2}-1$ and $\mu_{3,4} = \frac{1}{4}\left(3(n-2) \pm \sqrt{n(17n-32)}\right)$. Consequently, we can write

$$Spec_H(\Omega_{D_{2n}}) = \left\{ \left(\frac{1}{4}\left(3(n-2) + \sqrt{n(17n-32)}\right)\right)^1, \left(-\frac{3}{2}\right)^{\frac{n}{2}-1}, \left(-\frac{1}{2}\right)^{n-3+\frac{n}{2}}, \left(\frac{1}{4}\left(3(n-2) - \sqrt{n(17n-32)}\right)\right)^1 \right\},$$

and this proof is now complete.

□

We describe the spectral radius of $\Omega_{D_{2n}}$ based on the characteristic polynomial in Theorem 3.4.

Theorem 3.5. *In $\Omega_{D_{2n}}$, the Harary spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_H(\Omega_{D_{2n}}) = \begin{cases} \frac{1}{4}\left(3n-4+\sqrt{n(17n-16)}\right), & \text{if } n \text{ is odd} \\ \frac{1}{4}\left(3(n-2)+\sqrt{n(17n-32)}\right), & \text{if } n \text{ is even} \end{cases}.$$

Proof. Based on Theorem 3.4 and Definition 7, we obtain the spectral radius of $H(\Omega_{D_{2n}})$ as the maximum of μ_i, for $i = 1, 2, 3, 4$ and we complete the proof. □

We obtain the energy of $\Omega_{D_{2n}}$ based on the Harary matrix of $\Omega_{D_{2n}}$.

Theorem 3.6. *The Harary energy of* $\Omega_{D_{2n}}$ *is*

(1) *for* n *is odd:* $\varepsilon_H(\Omega_{D_{2n}}) = \frac{3}{2}(3n-4)$
(2) *for* n *is even:* $\varepsilon_H(\Omega_{D_{2n}}) = \frac{3}{2}(3n-6)$.

Proof. (1) Based on Definition 6 and Theorem 3.4 (1) for odd n and $n = 3$, we have two different eigenvalues. Then $\varepsilon_H(\Omega_{D_{2n}}) = 2\sqrt{6}$. For the $n > 3$ case, the Harary energy of $\Omega_{D_{2n}}$ is computed as given below:

$$\begin{aligned}\varepsilon_H(\Omega_{D_{2n}}) =&(n-2)\left|-\frac{1}{2}\right| + (n-1)\left|-1\right| + \left|\frac{1}{4}\left(3n-4 \pm \sqrt{n(17n-16)}\right)\right| \\ =&\frac{3}{2}(3n-4).\end{aligned}$$

(2) Suppose that n is even and according to Definition 6 and Theorem 3.1 (2), we get the distance energy of $\Omega_{D_{2n}}$:

$$\begin{aligned}\varepsilon_H(\Omega_{D_{2n}}) =& \left(\frac{n}{2}-1\right)\left|-\frac{3}{2}\right| + \left(n-3+\frac{n}{2}\right)\left|-\frac{1}{2}\right| + \left|\frac{1}{4}\left(3(n-2) - \sqrt{n(17n-32)}\right)\right| \\ =&\frac{3}{2}(3n-6).\end{aligned}$$

□

3.3. **Distance Laplacian Energy.** This subsection presents the distance Laplacian matrix of $\Omega_{D_{2n}}$.

Theorem 3.7. *In* $\Omega_{D_{2n}}$*, the spectrum of distance Laplacian matrix of* $\Omega_{D_{2n}}$ *is*

(1) *for* n *is odd:* $Spec_{DL}(\Omega_{D_{2n}}) = \left\{(3n-2)^{n-2}, (2n-1)^n, (0)^1\right\}$.
(2) *for* n *is even:* $Spec_{DL}(\Omega_{D_{2n}}) = \left\{(3n-4)^{n-3}, (2(n-1))^{\frac{n}{2}}, (2n)^{\frac{n}{2}}, (0)^1\right\}$.

Proof. (1) Suppose n is odd. Referring to Definition 4, Theorem 2.2 and Theorem 2.1 (1), we can set the distance Laplacian matrix of $\Omega_{D_{2n}}$ of size $(2n-1) \times (2n-1)$ by excluding the central element D_{2n}. The entries of $DL(\Omega_{D_{2n}}) = [b_{pq}]$ can be listed as follows.

(a) for distinct indices $p, q \in \{1, 2, \ldots, n-1\}$, the matrix entries satisfy $b_{pq} = 0 - 2 = -2$;
(b) for $p \in \{1, 2, \ldots, n-1\}$ and $q \in \{n, n+1, \ldots, 2n-1\}$, or with the indices interchanged, we have $b_{pq} = 0 - 1 = -1$;
(c) for distinct indices $p, q \in \{n, n+1, \ldots, 2n-1$, the corresponding entries are $b_{pq} = 0 - 1 = -1$;
(d) the entries $b_{pq} = (3n-4) - 0 = 3n-4$ whenever $p = q$ and $1 \leq p, q \leq n-1$;
(e) for diagonal indices $p = q \in \{n, n+1, \ldots, 2n-1\}$, the corresponding entries are $b_{pq} = 2(n-1) - 0 = 2(n-1)$.

Then $DL(\Omega_{D_{2n}})$ is as follows:

$$DL(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 3n-4 & -2 & \dots & -2 & -1 & -1 & \dots & -1 \\ -2 & 3n-4 & \dots & -2 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & 3n-4 & -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & 2(n-1) & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & -1 & 2(n-1) & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 & -1 & -1 & \dots & 2(n-1) \end{pmatrix} \end{array}.$$

The matrix above can be written in block matrix form as follows.

$$DL(\Omega_{D_{2n}}) = \begin{pmatrix} -2J_{n-1} + (3n-2)I_{n-1} & -J_{(n-1)\times n} \\ -J_{(n-1)\times n} & -J_n + (2n-1)I_n \end{pmatrix}.$$

By Theorem 2.3, with $a = 3n-4$, $b = 1$, $c = 2(n-1)$, $d = 2$, $n_1 = n-1$ and $n_2 = n$, we obtain

$$P_{DL(\Omega_{D_{2n}})}(\mu) = \mu(\mu - (3n-2))^{n-2}(\mu - (2n-1))^n.$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = 3n-2$ having multiplicity $n-2$, $\mu_2 = 2n-1$ having multiplicity n, and $\mu_3 = 0$ of multiplicity 1. Hence, the spectrum of $DL(\Omega_{D_{2n}})$ is

$$Spec_{DL}(\Omega_{D_{2n}}) = \left\{(3n-2)^{n-2}, (2n-1)^n, (0)^1\right\}.$$

(2) For the even n case, based on Definition 4, Theorem 2.2 and Theorem 2.1 (1), $DL(\Omega_{D_{2n}}) = [b_{ij}]$ is $(2n-2)\times(2n-2)$ by excluding the central element D_{2n}. The entries of $DL(\Omega_{D_{2n}})$ are
   (a) when $p, q = 1, 2, \dots, n-2$ and $p \neq q$, $b_{pq} = 0 - 2 = -2$;
   (b) The entries $b_{pq} = 0-1 = -1$ whenever one index lies in $\{1, 2, \dots, n-2\}$ and the other in $\{n-1, n, n+1, \dots, 2n-2\}$;
   (c) The matrix entries $b_{pq} = 0 - 2 = -2$ whenever $p = n-2+p$ and $q = n-2+\frac{n}{2}+r$ for some $r = 1, 2, \dots, \frac{n}{2}$, in either order;
   (d) for distinct $p, q \in \{n-1, n, n+1, \dots, 2n-2\}$, the entries satisfy $b_{pq} = 0 - 1 = -1$ except when $p = n-2+r$ and $q = n-2+\frac{n}{2}+r$ for some $r = 1, 2, \dots, \frac{n}{2}$, or with the indices interchanged;
   (e) for diagonal positions with $p = q \in \{1, 2, \dots, n-2\}$, the corresponding entries are $b_{pq} = 3(n-2) - 0 = 3(n-2)$;
   (f) for diagonal entries indexed by $p = q \in \{n-1, n, \dots, 2n-2\}$, the corresponding values are $b_{pq} = 2(n-1) - 0 = 2(n-1)$.

This implies that $DL(\Omega_{D_{2n}})$ is

$$\begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ a^{\frac{n}{2}+1}b \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c} \begin{array}{ccccccccccccc} a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \dots & a^{n-1}b \end{array} \\
\begin{pmatrix}
3(n-2) & -2 & \dots & -2 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
-2 & 3(n-2) & \dots & -2 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
-2 & -2 & \dots & 3(n-2) & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
-1 & -1 & \dots & -1 & 2(n-1) & -1 & \dots & -1 & -2 & -1 & \dots & -1 \\
-1 & -1 & \dots & -1 & -1 & 2(n-1) & \dots & -1 & -1 & -2 & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
-1 & -1 & \dots & -1 & -1 & -1 & \dots & 2(n-1) & -1 & -1 & \dots & -2 \\
-1 & -1 & \dots & -1 & -2 & -1 & \dots & -1 & 2(n-1) & -1 & \dots & -1 \\
-1 & -1 & \dots & -1 & -1 & -2 & \dots & -1 & -1 & 2(n-1) & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
-1 & -1 & \dots & -1 & -1 & -1 & \dots & -2 & -1 & -1 & \dots & 2(n-1)
\end{pmatrix} \end{array}.$$

It is a nine-block matrix as given below:

$$DL(\Omega_{D_{2n}}) = \begin{bmatrix} (3n-4)I_{n-2} - 2J_{n-2} & -J_{(n-2)\times\frac{n}{2}} & -J_{(n-2)\times\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (2n-1)I_{\frac{n}{2}} - J_{\frac{n}{2}} & -(J+I)_{\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & -(J+I)_{\frac{n}{2}} & (2n-1)I_{\frac{n}{2}} - J_{\frac{n}{2}} \end{bmatrix}.$$

Applying Theorem 2.4 with the choice $a = 3(n-2)$, $b = 2(n-1)$, $c = 1$, $d = 2$, then we derive

$$P_{DL(\Omega_{D_{2n}})}(\mu) = \mu\,(\mu - 3n + 4)^{n-3}\,(\mu - 2(n-1))^{\frac{n}{2}}\,(\mu - 2n)^{\frac{n}{2}}.$$

The eigenvalues of $\Omega_{D_{2n}}$ are precisely the zeros of $P_{DL(\Omega_{D_{2n}})}(\lambda)$. They are $\mu_1 = 0$ of multiplicity 1, $\mu_2 = 3n-4$ having multiplicity $n-3$, $\mu_3 = 2(n-1)$ of multiplicity $\frac{n}{2}$, and $\mu_4 = 2n$ of multiplicity $\frac{n}{2}$. Therefore,

$$Spec_{DL}(\Omega_{D_{2n}}) = \left\{(3n-4)^{n-3}, (2(n-1))^{\frac{n}{2}}, (2n)^{\frac{n}{2}}, (0)^1\right\}.$$

□

We present the spectral radius of $\Omega_{D_{2n}}$ according to the DL-matrix.

Theorem 3.8. *In $\Omega_{D_{2n}}$, the DL-spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_{DL}(\Omega_{D_{2n}}) = \begin{cases} 3n-2, & \text{if } n \text{ is odd} \\ 3n-4, & \text{if } n \text{ is even} \end{cases}.$$

Proof. By the spectrum in Theorem 3.7, the DL-spectral radius of $\Omega_{D_{2n}}$ is the maximum absolute of its eigenvalues. The proof is obvious. □

The distance Laplacian energy can be proved based on the spectrum of $DL(\Omega_{D_{2n}})$ as follows.

Theorem 3.9. *In $\Omega_{D_{2n}}$, the distance Laplacian energy of $\Omega_{D_{2n}}$ is*

$$\varepsilon_{DL}(\Omega_{D_{2n}}) = \begin{cases} (5n-4)(n-1), & \text{if } n \text{ is odd} \\ 5n^2 - 14n + 12, & \text{if } n \text{ is even} \end{cases}.$$

Proof. (1) Suppose n is odd. Referring to the spectrum in Theorem 3.7, the DL-energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned} \varepsilon_{DL}(\Omega_{D_{2n}}) =& (n-2)|3n-2| + (n)|2n-1| + (1)\,|0| \\ =& (5n-4)(n-1) \end{aligned}$$

(2) For the even n case, we derive the DL-energy of $\Omega_{D_{2n}}$ as given below:

$$\begin{aligned}\varepsilon_{DL}(\Omega_{D_{2n}}) =&(1)\,|0| + (n-3)\,|3n-4| + \left(\frac{n}{2}\right)|2(n-1)| + \left(\frac{n}{2}\right)|2n| \\ =&5n^2 - 14n + 12.\end{aligned}$$

□

3.4. **Distance Signless Laplacian Energy.** This section focuses on the distance signless Laplacian matrix of $\Omega_{D_{2n}}$. The next result is the characteristic formula of $DSL(\Omega_{D_{2n}})$ and its spectrum.

Theorem 3.10. *In $\Omega_{D_{2n}}$, the spectrum of $DSL(\Omega_{D_{2n}})$ is*

(1) *for odd n,*

$$\begin{aligned}Spec_{DSL}(\Omega_{D_{2n}}) = \Bigg\{&\left(\frac{1}{2}(8n - 11 + \sqrt{8n^2 - 24n + 25})\right)^1, (3n-6)^{n-2}, (2n-3)^{n-1}, \\ &\left(\frac{1}{2}(8n - 11 - \sqrt{8n^2 - 24n + 25})\right)^1\Bigg\},\end{aligned}$$

(2) *for even n,*

$$\begin{aligned}Spec_{DSL}(\Omega_{D_{2n}}) = \Bigg\{&\left(4n - 7 + \sqrt{2n^2 - 12n + 25}\right)^1, \left(4n - 7 - \sqrt{2n^2 - 12n + 25}\right)^1, \\ &(3n-8)^{n-3}, (2(n-1))^{\frac{n}{2}-1}, (2n-4)^{\frac{n}{2}}\Bigg\}.\end{aligned}$$

Proof. (1) Suppose n is odd. Referring to Definition 5, Theorem 2.2 and Theorem 2.1 (1), we can set the distance signless Laplacian matrix of $\Omega_{D_{2n}}$ of size $(2n-1)\times(2n-1)$ by excluding the central element D_{2n}. The entries of $DSL(\Omega_{D_{2n}}) = [c_{pq}]$ can be listed as follows.

(a) the value $c_{pq} = 0 + 2 = 2$ whenever $p \neq q$ and $p, q \in \{1, 2, \ldots, n-1\}$;
(b) for $p \in \{1, 2, \ldots, n-1\}$ and $q \in \{n, n+1, \ldots, 2n-1\}$, or coversely, then $c_{pq} = 0 + 1 = 1$;
(c) if $n \leq p, q \leq 2n-1$ with $p \neq q$, then $c_{pq} = 0 + 1 = 1$;
(d) for diagonal indices $p = q \in \{1, 2, \ldots, n-1\}$, the corresponding entries are $c_{pq} = (3n-4) + 0 = 3n-4$;
(e) for diagonal indices $p = q \in \{n, n+1, \ldots, 2n-1\}$, then $c_{pq} = 2(n-1) + 0 = 2(n-1)$.

Then $DSL(\Omega_{D_{2n}})$ is as follows:

$$DSL(\Omega_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array}\begin{array}{c} \begin{array}{cccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 3n-4 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 \\ 2 & 3n-4 & \ldots & 2 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \ldots & 3n-4 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 2(n-1) & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 1 & 2(n-1) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 2(n-1) \end{pmatrix} \end{array}.$$

The preceding matrix admits the following block decomposition.

$$DSL(\Omega_{D_{2n}}) = \begin{pmatrix} 2J_{n-1} + (3n-6)I_{n-1} & J_{(n-1)\times n} \\ J_{(n-1)\times n} & J_n + (2n-3)I_n \end{pmatrix}.$$

By Theorem 2.3, with $a = 3n-4$, $b = -1$, $c = 2(n-1)$, $d = -2$, $n_1 = n-1$ and $n_2 = n$, we obtain

$$P_{DSL(\Omega_{D_{2n}})}(\mu) = (\mu-(3n+6))^{n-2}(\mu-(2n-3))^{n-1}\left(\mu^2 - (8n-11)\mu + 2(n-1)(7n-12)\right).$$

The eigenvalues of $\Omega_{D_{2n}}$ are $\mu_1 = 3n-6$ of multiplicity $n-2$, $\mu_2 = 2n-3$ of multiplicity $n-1$, and $\mu_{3,4} = \frac{1}{2}(8n-11 \pm \sqrt{8n^2-24n+25})$ each of multiplicity 1. Hence, the spectrum of $SDL(\Omega_{D_{2n}})$ is

$$Spec_{DSL}(\Omega_{D_{2n}}) = \left\{ \left(\frac{1}{2}(8n-11+\sqrt{8n^2-24n+25})\right)^1, (3n-6)^{n-2}, (2n-3)^{n-1}, \left(\frac{1}{2}(8n-11-\sqrt{8n^2-24n+25})\right)^1 \right\}.$$

(2) For the even n case, according to Definition 5, Theorem 2.2 and Theorem 2.1 (1), $DSL(\Omega_{D_{2n}}) = [c_{pq}]$ is $(2n-2)\times(2n-2)$ by excluding the central element D_{2n}. The entries of $DSL(\Omega_{D_{2n}})$ are
  (a) the matrix elements $c_{pq} = 0+2 = 2$ whenever $p \neq q$ and $p, q \in \{1, 2, \ldots, n-2\}$;
  (b) the values $c_{pq} = 0+1 = 1$ whenever one index belongs to $\{1, 2, \ldots, n-2\}$ and the other to $\{n-1, n, n+1, \ldots, 2n-2\}$;
  (c) the matrix elements $c_{pq} = 0+2 = 2$ whenever $p = n-2+r$ and $q = n-2+\frac{n}{2}+r$ for some $r = 1, 2, \ldots, \frac{n}{2}$, in either order;
  (d) the value $c_{pq} = 0+1 = 1$ for all unequal indices $p, q \in \{n-1, n, n+1, \ldots, 2n-2\}$, apart from the index pairs $(p = n-2+r,\ q = n-2+\frac{n}{2}+r)$ with $r = 1, 2, \ldots, \frac{n}{2}$, and their reversals;
  (e) for diagonal indices $p = q \in \{1, 2, \ldots, n-2\}$, then $c_{pq} = 3(n-2)+0 = 3(n-2)$;
  (f) for diagonal indices $p = q \in \{n-1, , \ldots, 2n-2\}$, the corresponding entries are $c_{pq} = 2(n-1)+0 = 2(n-1)$.

This implies that $DL(\Omega_{D_{2n}})$ is

$$\begin{array}{c|cccccccccccc}
 & a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \ldots & a^{n-1}b \\
\hline
a & 3(n-2) & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
a^2 & 2 & 3(n-2) & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1} & 2 & 2 & \ldots & 3(n-2) & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
b & 1 & 1 & \ldots & 1 & 2(n-1) & 1 & \ldots & 1 & 2 & 1 & \ldots & 1 \\
ab & 1 & 1 & \ldots & 1 & 1 & 2(n-1) & \ldots & 1 & 1 & 2 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{\frac{n}{2}-1}b & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 2(n-1) & 1 & 1 & \ldots & 2 \\
a^{\frac{n}{2}}b & 1 & 1 & \ldots & 1 & 2 & 1 & \ldots & 1 & 2(n-1) & 1 & \ldots & 1 \\
a^{\frac{n}{2}+1}b & 1 & 1 & \ldots & 1 & 1 & 2 & \ldots & 1 & 1 & 2(n-1) & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
a^{n-1}b & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 2 & 1 & 1 & \ldots & 2(n-1)
\end{array}.$$

It is a nine-block matrix as given below:

$$DSL(\Omega_{D_{2n}}) = \begin{bmatrix} (3n-8)I_{n-2}+2J_{n-2} & J_{(n-2)\times\frac{n}{2}} & J_{(n-2)\times\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (2n-3)I_{\frac{n}{2}}+J_{\frac{n}{2}} & (J+I)_{\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J+I)_{\frac{n}{2}} & (2n-3)I_{\frac{n}{2}}+J_{\frac{n}{2}} \end{bmatrix}.$$

By Theorem 2.4 taking $a = 3(n-2)$, $b = 2(n-1)$, $c = -1$, $d = -2$, then we derive

$$P_{DSL(\Omega_{D_{2n}})}(\mu) = (\mu - 3n + 8)^{n-3} (\mu - 2(n-1))^{\frac{n}{2}-1} (\mu - 2n + 4)^{\frac{n}{2}} (\mu^2 - (8n-14)\mu + 14n^2 - 44n + 24).$$

The eigenvalues of $\Omega_{D_{2n}}$ are precisely the zeros of $P_{DSL(\Omega_{D_{2n}})}(\lambda)$. They are $\mu_1 = 3n-8$ of multiplicity $n-3$, $\mu_2 = 2(n-1)$ having multiplicity $\frac{n}{2}-1$, and $\mu_3 = 2n-4$ having multiplicity $\frac{n}{2}$, and $\mu_{4,5} = 4n-7 \pm \sqrt{2n^2-12n+25}$. Thus, the spectrum of $DSL(\Omega_{D_{2n}})$ is

$$Spec_{DSL}(\Omega_{D_{2n}}) = \left\{ \left(4n-7+\sqrt{2n^2-12n+25}\right)^1, \left(4n-7-\sqrt{2n^2-12n+25}\right)^1, (3n-8)^{n-3}, (2(n-1))^{\frac{n}{2}-1}, (2n-4)^{\frac{n}{2}} \right\}.$$

□

We obtain the spectral radius of $\Omega_{D_{2n}}$ according to the result of the previous theorem.

Theorem 3.11. *In $\Omega_{D_{2n}}$, the DSL-spectral radius of $\Omega_{D_{2n}}$ is*

$$\rho_{DSL}(\Omega_{D_{2n}}) = \begin{cases} \frac{1}{2}(8n-11+\sqrt{8n^2-24n+25}), & \text{if } n \text{ is odd} \\ 4n-7+\sqrt{2n^2-12n+25}, & \text{if } n \text{ is even} \end{cases}.$$

Proof. By Theorem 3.10, we have the DSL-spectral radius of $\Omega_{D_{2n}}$ by taking the maximum absolute of eigenvalues. □

We now deliver the energy of $\Omega_{D_{2n}}$ associated with the DSL-matrix.

Theorem 3.12. *In $\Omega_{D_{2n}}$, the DSL-energy of $\Omega_{D_{2n}}$ is*

$$\varepsilon_{DSL}(\Omega_{D_{2n}}) = \begin{cases} 5n^2-7n+15, & \text{if } n \text{ is odd} \\ 5n^2-14n+12, & \text{if } n \text{ is even} \end{cases}.$$

Proof. (1) Suppose n is odd. Based on Theorem 3.10, the distance signless Laplacian energy of $\Omega_{D_{2n}}$ is

$$\varepsilon_{DSL}(\Omega_{D_{2n}}) = (n-2)|3n-6| + (n-1)|2n-3| + \left|\frac{1}{2}(8n-11 \pm \sqrt{8n^2-24n+25})\right| = 5n^2-7n+15$$

(2) For the even n case, by Theorem 3.10, then

$$\varepsilon_{DSL}(\Omega_{D_{2n}}) = (n-3)\,|3n-8| + \left(\frac{n}{2}-1\right)|2(n-1)| + \left(\frac{n}{2}\right)|2n-4| + \left|4n-7 \pm \sqrt{2n^2-12n+25}\right| = 5n^2-14n+12.$$

□

4. Further Discussion

Further discussion from Theorem 3.3, 3.6, 3.9, and 3.12, we observe the classification of the energy of $\Omega_{D_{2n}}$.

Corollary 4.1. *$\Omega_{D_{2n}}$ is hyperenergetic associated with the distance, Harary, distance Laplacian, and distance signless Laplacian matrices.*

In addition, the results yield the following fact which complies with the result from [5] and [19].

Corollary 4.2. *The energy is always an even integer in $\Omega_{D_{2n}}$ for the distance matrix except for $n = 3$.*

Corollary 4.3. *The energy is always an even integer in $\Omega_{D_{2n}}$ for the distance Laplacian matrix.*

Corollary 4.4. *The energy is never an odd integer in $\Omega_{D_{2n}}$ with respect to Harary and distance signless Laplacian matrices.*

The relationships between the obtained energies are presented in the following two facts.

Corollary 4.5. *In $\Omega_{D_{2n}}$, we have $\varepsilon_D(\Omega_{D_{2n}}) < \varepsilon_H(\Omega_{D_{2n}}) < \varepsilon_{DL}(\Omega_{D_{2n}}) \leq \varepsilon_{DSL}(\Omega_{D_{2n}})$.*

Corollary 4.6. *In $\Omega_{D_{2n}}$, we have $\varepsilon_{DL}(\Omega_{D_{2n}}) = \varepsilon_{DSL}(\Omega_{D_{2n}})$ for even n.*

5. Conclusion

The energies of the non-commuting graph arising from the dihedral groups have been presented. The formula for distance-based energies has been formulated. In addition, this paper also shows the relationship between those energies and classifies the energies based on their value. For further research, extending the study to different groups is recommended.

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References

[1] A. Abdollahi, S. Akbari, and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra **298** 2 (2006) 468–492. https://doi.org/10.1016/j.jalgebra.2006.02.015

[2] A. Alhevaz, M. Baghipur, S. Pirzada, *On Distance Signless Laplacian Estrada Index and Energy of Graphs*, Kragujev. J. Math. **45** 6 (2021) 837–858. http://elib.mi.sanu.ac.rs/files/journals/kjm/68/kjmn68p837-858.pdf

[3] A. Alhevaz, M. Baghipur, H. Ramane, X. Li, *On the distance signless Laplacian Estrada index of graphs*, Asian-Eur. J. Math. **15** 4 (2022) 2250078. https://doi.org/10.1142/S1793557122500784

[4] M. Aouchiche, P. Hansen, *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. **439** (2013) 21–33. https://doi.org/10.1016/j.laa.2013.02.030

[5] R. B. Bapat, S. Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc. **1** (2004) 129–132. https://www.isid.ac.in/ rbb/paper11.pdf

[6] K. C. Das, I. Gutman, I. Milovanovic, E. Milovanovic, B. Furtula, *Degree-based energies of graphs*, Linear Algebra Appl. **554** (2018) 185–204. https://doi.org/10.1016/j.laa.2018.05.027

[7] E. Estrada, *Characterization of the amino acid contribution to the folding degree of proteins*, Proteins, **54** (2004) 727–737. https://doi.org/10.1002/prot.10609

[8] E. Estrada, *The Structure of Complex Networks-Theory and Applications*, New York, Oxford Univ. Press, 2012.

[9] S. Filipovski, R. Gutman, *Bounds for the energy of graphs*, Mathematics, **9** 1687 (2021) 1–10. https://doi.org/10.3390/math9141687

[10] I. Gutman, *The energy of graph*, Ber. Math. Statist. Sekt. Forschungszenturm Graz. **72** (1978) 1–2.

[11] R. A. Horn, C. A. Johnson, *Matrix analysis*, Cambridge, Cambridge University Press, 1985.

[12] A. D. Gungor, S. B. Bozkurt, *On the distance Estrada index of graphs*, Hacettepe J. Math. Stat. **38** 3 (2009) 277–283. https://dergipark.org.tr/en/download/article-file/86791

[13] H. Ibrahim, R. Sharafdini, T. Reti, A. Akwu, *Wiener-Hosoya matrix of connected graphs*, Mathematics, **9** 359 (2021) 1–12. https://doi.org/10.3390/math9040359

[14] G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60** 2 (2008) 461–472.

[15] O. Ivanciuc, T. S. Balaban, A. T. Balaban, *Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices*, J. Math. Chem. **12** (1993) 309–318. https://doi.org/10.1007/BF01164642

[16] A. Jahanbani, H. Shooshtari, M. A. Ul Haq, S. Pirzada, *On the eigenvalues of the distance signless Laplacian matrix of graphs*, Proyecciones J. Math. **43** 5 (2024) 1253–1267. https://doi.org/10.22199/issn.0717-6279-6247

[17] X. Li, Y. Shi, I. Gutman, *Graph energy*, New York, Springer, 2012.

[18] S. Pirzada, I. Gutman, *Energy of a graph is never the square root of an odd integer*, Appl. Anal. Discrete Math. **2** (2008) 118–121. https://pefmath.etf.bg.ac.rs/vol2num1/AADM-Vol2-No1-118-121.pdf

[19] S. Pirzada, M. A. U. Haq, *On the spread of the distance signless Laplacian matrix of a graph*, Acta Univ. Sapientiae, Inform. **15** 1 (2023) 38–45. https://doi.org/10.2478/ausi-2023-0004

[20] H. S. Ramane, S. S. Shinde, *Degree exponent polynomial of graphs obtained by some graph operations*, Electron. Notes Discrete Math. **63** (2017) 161–168. https://doi.org/10.1016/j.endm.2017.11.010

[21] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, A. Al-Quran, *Closeness Energy of Non-Commuting Graph for Dihedral Groups*, Eur. J. Pure Appl. Math. **17** 1 (2024) 212–221. https://doi.org/10.29020/nybg.ejpam.v17i1.5020

[22] M. U. Romdhini, A. Nawawi, F. Al-Sharqi, A. Al-Quran, S. R. Kamali, *Wiener-Hosoya energy of non-commuting graph for dihedral groups*, Asia Pac. J. Math. **11** 9 (2024) 1–9. https://doi.org/10.28924/APJM/11-9

[23] M. U. Romdhini, Abdurahim, A. E. S. H. Maharani, A. Nawawi, F. Al-Sharqi, I. H. Dani, *Closeness Laplacian and Closeness signless Laplacian Energies of Non-Commuting Graph for Dihedral Groups*, Ann. Acad. Rom. Sci. Ser. Math. Appl. **18** 2 (2026) 1–17.

Mamika Ujianita Romdhini
Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia
E-mail address: mamika@unram.ac.id

Athirah Nawawi
Department of Mathematics and Statistics, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
E-mail address: athirah@upm.ac

Faisal Al-Sharqi
Department of Mathematics, Faculty of Education for Pure Sciences, University Of Anbar, Ramadi 55431, Anbar, Iraq.
College of Engineering, National University of Science and Technology, Dhi Qar, Iraq.
Department of Mathematics, College of Education, Al-Ayen Iraqi University, An Nasiriyah, Iraq
E-mail address: faisal.ghazi@uoanbar.edu.iq

Abdurahim
Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia
E-mail address: abdurahim@staff.unram.ac.id

Andika Ellena Saufika Hakim Maharani
Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia
E-mail address: a.ellena.saufika@staff.unram.ac.id

Ifan Hasnan Dani
Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia
E-mail address: ifanhasnandani@gmail.com